C. *Atomic Systems based on Free Electrons, positive and negative, and their Stability.* By R. HARGREAVES, *M.A.*[*]

PART I.—STEADY MOTION.

IN the atomic scheme of which a planetary system is the model, negative electrons have the position of planets, and a positive charge is condensed at a central nucleus. It is then postulated from experience, chemical, electrical, and spectroscopic, that the nucleus shall contain an integral number of standard charges. But if this postulate is essential, it seems imperative to provide a structure in which positive as well as negative electrons are discrete. The scheme with multiple core takes no account of the mutual repulsion of members of the core, and so ignores the primary conception of separate existence attaching to an integral number. An orbital motion, which the scheme provides, is however in all probability an essential feature of the atomic aggregate.

For the present scheme the two types of electron furnish the raw material ; they are assumed to be discrete, to carry charges differing only in sign, and to possess inertias differing widely in amount. It is found possible to realize kinetic structures in which these constituents are bound together by the electrical forces due to their mutual action alone. There is no need to suppose the normal laws of attraction and repulsion to be in abeyance,

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no need for a cement as difficult to explain as the atom itself; an orbital motion is provided, and the materials could not be fewer. The question how far these structures, regarded as representations of the atom, meet the demands from the various branches of Physics and Chemistry is therefore a fundamental one, to which it is desirable to obtain an answer as complete as possible.

§ 1. The characteristic feature of the scheme is a structure or configuration consisting of two concentric rings, one of positive the other of negative electrons in like number. The charges are disposed at regular intervals on the circumferences of their respective circles, while in angular position elements of one type lie half-way between those of opposite type (fig. 1). For this configuration we are seeking, not a

Fig. 1.
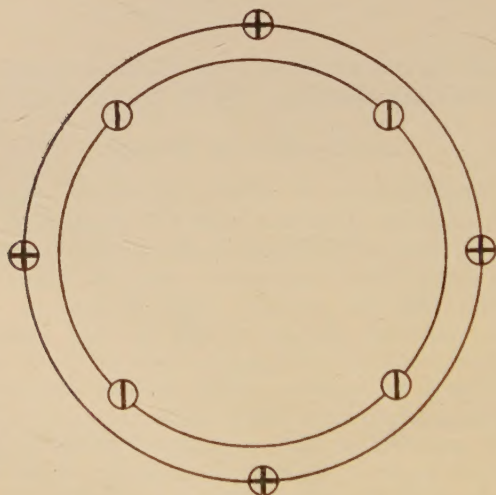
Arrangement of positive and negative electrons in concentric circles for $n=4$.

position of equilibrium but one of steady motion for each element in its own circle. The symmetry of the structure ensures a purely central force on each element; the maintenance of the symmetry demands a common angular velocity, which requires an adjustment of the radii. The condition to be fulfilled is

$$m_1\omega^2 a_1 : m_2\omega^2 a_2 = F_1 : F_2, \quad . \quad . \quad . \quad \text{(i.)}$$

m being mass, a radius, F attraction to centre, with the subscripts 1 and 2 attached respectively to positive and negative

elements. When account is taken of the fact that m_2/m_1 is of the order $1/1800$, it is clear that if the electrical forces $F_1 F_2$ are on the same scale, it will be necessary to compensate the greater mass of positive elements by a relatively small radius. The attempt to do this fails on account of the strong repulsive force called into play. If there is no great inequality of radii then the force on the negative element must be relatively small, that is we must seek a position which is nearly one of equilibrium for negative but not for positive elements. There is only one such position. In that position the negative ring has a less radius than the positive, with a difference small except for small values of the number n of elements in the ring, as will appear from the following argument, certainly applicable when n is not small.

Fig. 2.
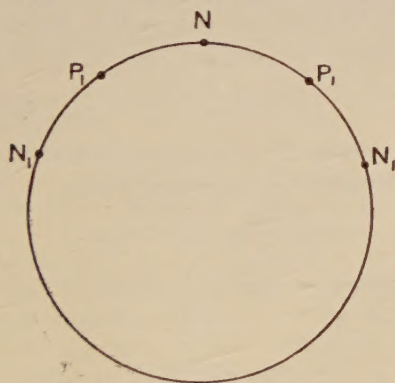
Suppose all charges on one circle as in fig. 2, and consider the action on a negative electron N. The positive pairs on either side give a diminishing series of contributions to a central attraction; the repulsions due to negative pairs form a second diminishing series. The first series is the greater as representing nearer pairs. If N were now placed in the middle of the chord $P_1 P_1$, the most important term of the first series would be out of action, and the second series representing repulsion would be the greater. Between these two positions of N is the position sought, which for n great is found to show a difference in radii about $2/3$ of the sagitta of the arc $P_1 P_1$. Rough values for the ratio of radii are 1.73 for $n = 2$, 1.19 for $n = 4$, 1.05 for $n = 8$; the departure from unity already small and pointing to the asymptotic law, variation as n^{-2}.

§ 2. We may for shortness use the term ion for "positive electron," leaving the unqualified word for the negative electron: the mobility of the latter would suggest an interchange of terms if it were feasible.

The feature that the outer ring contains the more massive ions contributes to stability through the protection afforded to the mobile electrons. In part this is due to actual screening, but the main defence lies in the fact that near each electron are two ions, so that external action on the former calls into play a strong counter-action from the latter.

The angular velocity in the orbit is known when the central force has been found, whether we are dealing with the double ring or a single ring with core ; but in the former case the adjustment of radii requires a preliminary calculation. In each case tabulation is needed when the number n in a ring is small, and an asymptotic formula can be used when n is not small. For the double ring when the adjustment of radii has been made, the angular velocity is given by

$$m_1 \omega^2 a_1{}^3 = N \epsilon^2, \quad . \quad . \quad . \quad . \quad . \quad \text{(ii. } a)$$

where ϵ is the standard charge and N a number tabulated below. An asymptotic formula (*cf.* § 13) is

$$N = \cdot 441\, n + \cdot 424\, n^{-1} + \ldots \quad . \quad . \quad . \quad \text{(ii. } b)$$

which is of service from $n = 10$ upwards.

For a multiple core $+ n\epsilon$, and a ring of n electrons the formula is

$$m_2 \omega^2 a^3 = N_c \epsilon^2, \quad . \quad . \quad . \quad . \quad \text{(iii. } a)$$

where

$$N_c = \cdot 98\, n - \cdot 36646\, n \log_{10} n + \cdot 02182\, n^{-1} \ldots \quad \text{(iii. } b)$$

is an asymptotic formula. The calculations required for N are more serious than for N_c, but the asymptotic formula simpler.

The value of n for which N_c becomes negative lies beyond the range of n required for application to atoms. N_c/n is at first greater than N/n, is not much different for n about 30, and then diminishes much more rapidly than N/n as n is increased. As (ii. a) and (iii. a) have the inertias m_1 and m_2 respectively it is clear that the numerical relations connecting ω and a are very different, and must involve corresponding differences of interpretation.

§ 3. It is consistent with the hypothesis of discrete charges to suppose a unit at the centre. The adjustment of radii for steady motion is possible with a positive centre when $n \geq 4$, not when $n = 2$ or 3, for the mutual repulsion is not then adequate to balance the central attraction, and so give a position of equilibrium near which an electron in the ring must lie. The inequality of radii is increased by the presence of the central unit. If the latter is displaced axially while the positive and negative rings are supposed to remain in one plane the position is clearly stable for the central charge, because the attractive elements are the nearer.

For a negative centre the adjustment of radii is possible for all values of n; for $n > 4$ the inequality of radii is reduced, for $n = 2$, 3, or 4 it is reversed and the negative ring is outside. With the two rings in one plane the position would, for axial displacement of the central unit, be stable for $n = 2$, 3, or 4, unstable for other cases. A more complete treatment in which account is taken of the relative movement of the planes of the two rings, *shows that the negative electron at the centre is always axially unstable, the positive always stable.* Cf. § 33.

A positive structure is found for the cases $n = 2$ or 3, somewhat unexpectedly, by taking two positive units on the axis, one above the other below the plane of the rings, with a separation wide enough to make the attraction of the pair on an electron of the ring less than that of a unit at the centre. This proves the beginning of a series which continues up to the value $n = 8$ and there stops. In the range from $n = 5$ to $n = 8$, containing important elements, B, C, N, O, there are two positions : one where the attraction of the pair on an electron is less than that of a unit at the centre, the other with closer axial units and an attraction greater than that of a unit at the centre.

This completes the forms of the fundamental structures, for which the symbols $R_n(0)$, $R_n(+)$, $R_n(+, +)$ may be used—rings with vacant centre, positive centre, or, in a few cases, two axial units.

§ 4. As subsidiary to the main structure it is proposed to consider the case of electrons describing nearly circular orbits about the centre of $R_n(+)$, either inside the inner or outside the outer ring. The central unit alone can maintain as many as four electrons in one circular orbit, with only a small residual attraction for four. The central component of force due to the ring fluctuates between the two signs, but its mean value gives a reinforcement of the central attraction ;

fluctuation and mean value being greatest for orbits near to the rings. No truly circular orbit is possible, and the inner and outer positions for satellites are in general alternative.

The cases of $R_n(+, +)$ from $n=5$ to $n=8$, in which the closer position is taken, have the advantage of an increased central attraction, for outer satellites certainly.

The presence of satellites alters in some measure the character of the two rings, which may now be regarded as a *type of freely equilibrated nucleus controlling a limited number of satellites.*

As regards the number of satellites there is little doubt that the case of one only is the most important on the ground of freedom from instability ; if internal its relation to the nucleus seems specially intimate. Outer satellites are more exposed to attack but also more readily replaced than inner satellites, when we consider the possible action of other atoms or of free electrons.

§ 5. The polarizing action of external force is simplest when the force is directed along the axis of the rings, and then brings about a small separation of the planes of positive and negative elements. A force in the plane of the rings gives rise to displacements radial and tangential so nearly balanced in opposite parts of a ring as to yield little resultant. The bond between the rings is sufficiently strong, at any rate when n is not small, to limit the separation of planes to an amount which does not sensibly modify the structure. For satellites the resistance offered to axial force is slight. Again a central ion, though stable axially, suffers a displacement on a sensibly greater scale than the separation of rings. If it is detached by external force it will be subject to attraction by the next neutral structure in its path, may be incorporated and so move forward by halting steps, functioning as centre for a succession of atoms. The mode of conduction differs from that by free electrons much as electrolytic from metallic conduction ; free electrons have large velocities in random directions modified by external force, while ions move with less velocity but follow more closely the action of external force. There is the further distinctive feature of incorporation at various stages. *Cf.* § 34.

An external field of magnetic force in the direction of the axis will give rise to central force in opposite senses on the elements of the two rings. According to the sense of the magnetic field *or* according to the direction of rotation in the rings, the effect may be to increase or diminish the difference of radii, thus altering the period.

The external magnetic forces due to the separate rings are not so nearly balanced as electrostatic forces, for there is a difference in velocity which implies a difference in the strengths of the equivalent continuous circuits, whereas for electrostatic force only the difference in position exists. If satellites are present there are unbalanced effects of much greater amount, different also for internal and external satellites ; they will be opposite in character to those of the double ring when the direction of revolution is the same, for in the latter the balance is in favour of the positive. To these differences we must look for an explanation of diamagnetic and paramagnetic properties.

In these two fundamental matters the presence of satellites is seen to be influential, and in respect to spectroscopic phenomena their importance is at once evident.

§ 6. It is proposed to bring forward material bearing on (i.) atomic weights, (ii.) the gravitational constant.

In respect to (i.), contact is sought with the scheme of atomic numbers suggested by van der Broek and developed by Moseley and Bohr. As for the neutral ring there are n ions of mass m_1 and n electrons of mass m_2, the primary measure of mass is $n(m_1 + m_2)$. Since $m_1 + m_2$ cannot differ much from the mass of the hydrogen atom, a varying factor somewhat greater than 2 is to be accounted for. Consider first the point of view of inertia.

(a) The equation for internal motion $N\epsilon^2 = m_1\omega^2 a_1^3$ may be written as $n\epsilon^2/a_1^2 = (nm_1/N)\omega^2 a_1 \equiv M_n\omega^2 a_1$, say ; a form in which simplicity is given to the member showing electrical action by modifying the measure of mass. Thus if we treat nM_n as mass of the double ring, and seek to identify it with actual atomic weight, we are supposing this modification of mass to apply to external as well as to internal relations. The modifying factor n/N by which we pass from m_1 to M_n ranges in value from 1·909 for $n = 2$ to an asymptotic value 2·266, and proceeds with regularity but not at uniform rate. The actual values of the quotient atomic weight ÷ atomic number are irregular, and for n large are in excess of the limit just given : for n not large the modification is on the requisite scale. If the atomic number is taken to be the number n defining the double ring, whether with or without centre, or again with or without satellites, there will be various values of N comprised in one group. For example, the case of $R_n(+)$ with 3 satellites would give a sensible reduction of N and a correspondingly increased value of the factor n/N. Irregularities would then be

significant in relation to the proportion of atoms with centre and to the number of satellites carried.

The above seems the natural interpretation of atomic number in relation to the present scheme : but if that number represented a number of ions the grouping of cases would be different.

(b) If we set up as an ideal the elimination of the gravitational constant, so that a form m_2^2/r may appear in gravitation as compared with ϵ^2/r in electrical action, what value is suggested for mass ? A flat rate of transformation to M_g, in lieu of the variable M_n used above, would require $\gamma M_g^2 = m_2^2$; or with $\gamma = 6\cdot67 \times 10^{-8}$, $M_g = 3872\,m_2$. With Millikan's ratio of $m_1 : m_2$ this would imply $M_g = 2\cdot1\,m_1$, or with the ratio suggested below $M_g = 2\cdot13\,m_1$, values in close agreement with the average in (a). This numerical relation seems significant, but has no connexion with any special atomic scheme, at any rate of an obvious character.

(c) In dealing with the above figures a point which engaged my attention was the closeness of $\gamma m_1^2/m_2^2$ to the number $\frac{1}{2}(\cdot44127)$, the limit of $N/2n$. An attempted interpretation runs thus:—the coefficient in $\dfrac{N}{2n}\left(\dfrac{n^2\epsilon^2}{a_1}\right)$, an expression for internal kinetic energy, approaches in its *asymptotic* form to equality with the coefficient in $\dfrac{\gamma(m_1+m_2)^2}{m_2^2} \times \left(\dfrac{n^2 m_2^2}{r}\right)$, a gravitational potential. The equality gives $m_1 + m_2 = 1819\,m_2$ or $m_1 = 1818\,m_2$. Millikan gives for the unit of the atomic scale (m_0 say), $\epsilon/m_0 = 9650\,\mathrm{V}$; which with $\epsilon/m_2 = 1\cdot767 \times 10^7 \times \mathrm{V}$ gives $m_0 = 1831\,m_2$. An assured connexion between m_1 and the mass of the hydrogen atom seems to be wanting to give certainty to the value of the ratio m_1/m_2.

The above value has been used wherever in the calculations the ratio occurs. Possibly my first impression in respect to the number was too favourable.

§ 7. In concluding this sketch of the configuration it may be pointed out, that though by the addition of a central ion and satellites some variety and complexity is given, yet the margin of choice is strictly limited. In this respect the contrast with a multiple point-core is noticeable. A core $+10$ can maintain any number of electrons from 1 to 20 in orbital motion in one circle, with a possible residue varying from $+9$ to -10. The electrons may be distributed in several circles with a further range of varieties. Whether this

multiplicity of options is to be regarded as a valuable asset or as a source of embarrassment may be left to the reader to decide. The loss of such large freedom of choice is the price paid for the nuclear cement provided. The nucleus is of an entirely different order, one of mass rather than of charge, and it is endowed with orbital motion and possibilities of internal oscillation. The whole structure—rings, central ion, and satellites—does not admit of any but a quite small residual charge. *Cf.* § 23.

We proceed to the mathematical theory on which this general account is based, so far as it relates to steady motion. A second part contains an investigation of the natural oscillations and of the associated question of stability, considered with reference to the present scheme and also that of a multiple core.

Mathematical Theory.

§ 8. Let there be n charges ϵ each with mass m_1 on a circle of radius a_1, and n charges $-\epsilon$ each with mass m_2 on a circle of radius a_2, disposed as above.

For uniform motion in a circle we have

$$m_1\omega^2 a_1 = F_1, \qquad m_2\omega^2 a_2 = F_2. \quad . \quad . \quad . \quad (1)$$

When $F_1\,F_2$ are expressed in terms of the radii, the problem is to determine the ratio $x = a_1 : a_2$ so as to satisfy

$$F_2/a_2 = m_2/m_1 \times F_1/a_1. \quad . \quad . \quad . \quad . \quad (2)$$

With ratio known the forces can be expressed in terms of one linear magnitude. Since $m_2 : m_1 \equiv \mu$ is of order $1 : 1800$ a first approximation is got by determining x from $F_2 = 0$, and then finding ω from

$$m_1\omega^2 a_1 = F_1 \quad . \quad . \quad . \quad . \quad . \quad (3)$$

with the special value substituted in F_1. The central repulsive force on one charge in a circle of n charges is given by

$$\left. \begin{array}{c} \epsilon^2 c_n/2a^2 \\ \text{where} \qquad c_n = \tfrac{1}{2} \sum\limits_{s=1}^{n-1} \operatorname{cosec} s\pi/n. \end{array} \right\} \quad . \quad . \quad . \quad . \quad (4)$$

The attractive force on $-\epsilon$ due to the n charges $+\epsilon$ of the circle a_1 is given by

$$\sum_{r=0}^{n-1} \frac{\epsilon^2\{a_2 - a_1\cos(2r+1)\pi/n\}}{\{a_1^2 + a_2^2 - 2a_1 a_2\cos(2r+1)\pi/n\}^{3/2}}. \quad . \quad (5)$$

On introducing the ratio x equations (1) become

$$m_1\omega^2 a_1{}^3 = \epsilon^2\left\{f\!\left(\frac{1}{x}\right) - \tfrac{1}{2}c_n\right\}, \qquad m_2\omega^2 a_2{}^3 = \epsilon^2\{f(x) - \tfrac{1}{2}c_n\};$$

$$\cdots \quad (6\,a)$$

in which
$$f(x) = \sum_{r=0}^{n-1} \frac{1 - x\cos\,(2r+1)\pi/n}{\{1 + x^2 - 2x\cos\,(2r+1)\pi/n\}^{3/2}}. \quad (7)$$

If there is a positive charge at the centre, formulæ $(6\,a)$ are replaced by

$$m_1\omega^2 a_1{}^3 = \epsilon^2\left\{f\!\left(\frac{1}{x}\right) - \tfrac{1}{2}c_n - 1\right\}, \quad m_2\omega^2 a_2{}^3 = \epsilon^2\{f(x) - \tfrac{1}{2}c_n' + 1\}.$$

$$\cdots \quad (6\,b)$$

When the value of x satisfying $f(x) = \tfrac{1}{2}c_n$ is substituted in $f\!\left(\dfrac{1}{x}\right) - \tfrac{1}{2}c_n$ we have the tabulated number N, while N_μ is used for the corrected value when $m_2 : m_1$ is not neglected.

The kinetic energy of orbital motion is given by

$$2T = n(m_1 a_1{}^2 + m_2 a_2{}^2)\,\omega^2 = \frac{nN_\mu\epsilon^2}{a_1}\left(1 + \frac{\mu}{x^2}\right), \quad (8\,a)$$

for which it is generally sufficient to write

$$2T = nN\epsilon^2/a_1. \quad \cdots \quad (8\,b)$$

The potential energy U, total energy of orbital motion T, and total angular momentum H are then given by

$$E = T + U, \quad 2T + U = 0, \quad 2T = H\omega; \quad (9\,a)$$

while

$$2TH^2 = m_1 N^2 n^3 \epsilon^4 \quad \cdots \quad (9\,b)$$

is a relation into which neither ω nor a enters, written in the form suited to the approximation $(8\,b)$. A symmetrical form can be given to $(8\,a)$ and $(9\,b)$ by using

$$M = m_1 + m_2, \qquad Ma_s{}^2 = m_1 a_1{}^2 + m_2 a_2{}^2$$

and
$$N_s : N_\mu = Ma_s{}^3 : m_1 a_1{}^3.$$

§ 9. As the numerical work of solution only appears in the tabulated results, it may be of service to show the work for the simplest case $n = 2$. With $a_2 = 1$, $a_1 = x$ the equations are

$$m_2\omega^2 = \frac{2\epsilon^2}{(1 + x^2)^{3/2}} - \frac{\epsilon^2}{4}, \qquad m_1\omega^2 x = \frac{2x\epsilon^2}{(1 + x^2)^{3/2}} - \frac{\epsilon^2}{4x^2},$$

and x is to be found from

$$\frac{2}{(1+x^2)^{3/2}} - \frac{1}{4} = \mu \left\{ \frac{2}{(1+x^2)^{3/2}} - \frac{1}{4x^3} \right\}.$$

The first approximation gives $(1+x^2)^{3/2} = 8$ or $x = \sqrt{3}$, and for a second we have

$$\frac{2}{(1+x^2)^{3/2}} - \frac{1}{4} = \frac{\mu}{4}\left(1 - \frac{1}{x^3}\right) = \frac{\mu}{4}\left(1 - \frac{1}{3\sqrt{3}}\right).$$

With

$$x^2 = 3 - \kappa, \qquad \frac{3\kappa}{8} = \mu\left(1 - \frac{1}{3\sqrt{3}}\right),$$

$$x = \sqrt{3}\left(1 - \frac{\kappa}{6}\right) = \sqrt{3} - \frac{4\mu}{27}\left(3\sqrt{3} - 1\right),$$

and

$$x^3 = 3\sqrt{3} - \frac{4\mu}{3}\left(3\sqrt{3} - 1\right).$$

Thus

$$N_\mu = x^3\left[\frac{2}{(1+x^2)^{3/2}} - \frac{1}{4} + \frac{1}{4}\left(1 - \frac{1}{x^3}\right)\right] = \frac{x^3 - 1}{4} + \frac{x^3\mu}{4}\left(1 - \frac{1}{3\sqrt{3}}\right)$$

$$= \frac{3\sqrt{3} - 1}{4} - \frac{\mu}{12}(3\sqrt{3} - 1) = 1\cdot04904 - \cdot00019$$

with $\mu = 1 : 1818$;

while $x_\mu = \cdot73205 - \cdot00034$. In each case the correction for μ is quite small.

Table I. gives values of x and N for a vacant centre, Table II. values of x' and N' for a positive centre, obtained by direct solution of $(6\,a, b)$.

For the problem of two ions placed axially two variables are needed : x as before, and y where $2y$ is the ratio of the mutual distance of the ions to the radius of the negative ring. If we take the radius of this ring as 1, the nature of the change in the equations may be shown by writing them for $n = 4$. They are

$$\left.\begin{aligned}
\frac{1}{4y^2} + \frac{4y}{(x^2+y^2)^{3/2}} &= \frac{4y}{(1+y^2)^{3/2}}, \\[6pt]
\frac{2-x\sqrt{2}}{(1+x^2-x\sqrt{2})^{3/2}} + \frac{2+x\sqrt{2}}{(1+x^2+x\sqrt{2})^{3/2}} + \frac{2}{(1+y^2)^{3/2}} & \\
&= \cdot9571, \\[6pt]
\frac{x^2(2x-\sqrt{2})}{(1+x^2-x\sqrt{2})^{3/2}} + \frac{x^2(2x+\sqrt{2})}{(1+x^2+x\sqrt{2})^{3/2}} - \frac{2x}{(x^2+y^2)^{3/2}} & \\
&- \cdot9571 = N.
\end{aligned}\right\} \quad (10)$$

The first equation is the condition for equilibrium of the axial ions, the second that for the equilibrium of electrons in the ring (μ correction neglected), and the third gives the characteristic number on which the angular velocity depends.

§ 10. The position as regards a possible ambiguity in the solution for x may be explained in connexion with the graph (fig. 3) of $f(x)$ here drawn for $n=6$. The curve is typical

Fig. 3.
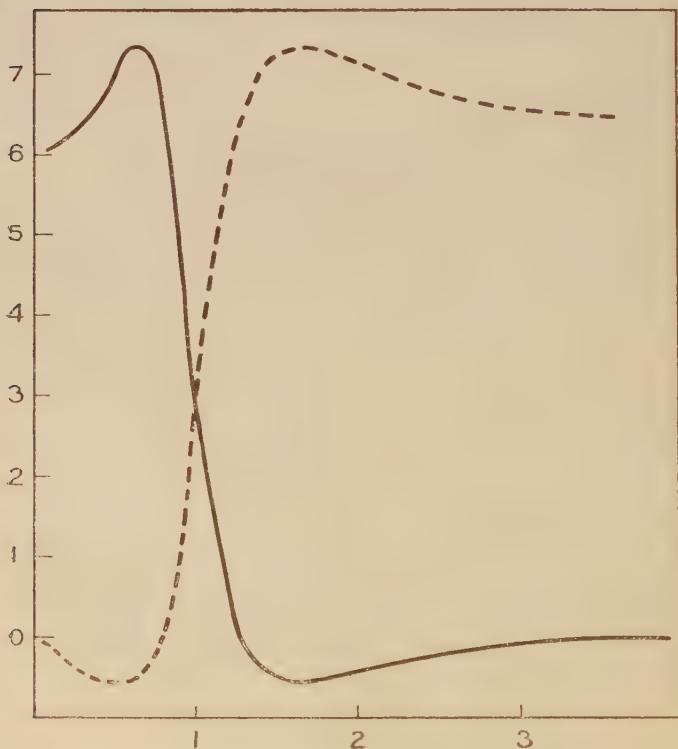
Graphs of $f(x)$ and $f\left(\dfrac{1}{x}\right)$, the latter dotted, for $n=6$.

of cases from $n=4$ upwards ; for $n=2$ or 3 the curve lies above the asymptote. The graph shows a minimum value n for $x=0$, followed by a maximum and a rapid descent to a minimum of small negative value, from which it rises to the asymptotic zero value.

Two solutions are possible : (i.) when the value assigned to $f(x)$ is greater than n and less than the maximum, and (ii.) when the assigned value is negative and greater than the second minimum. Case (i.) requiring $c_4 > 2n$ can only occur if $n > 473$, a number outside the range considered. Case (ii.) does not occur with vacant centre ; but occurs once, viz. with $n = 4$ for a positive centre. Here $\frac{1}{2}c_4 = \cdot 25 + \cdot 707 = \cdot 957$, and the assigned value of $f(x)$ is $- \cdot 043$, lying between 0 and the minimum $- \cdot 059$, which occurs near $x = 2 \cdot 39$. The second solution, on the up grade of the curve, corresponds to a position of instability.

The comparative straightness of the graph near $x = 1$ suggests a method of approximation to the value of $f(x)$ when n is great and $x - 1$ small, which gives at once the asymptotic value for n great and a serviceable approximation for moderate values of n. The solution of (6) by numerical test needed for small values of n becomes laborious between 10 and 20, and for prolonging tables beyond this range the use of such a method as is proposed is indispensable.

§ 11. With $x - 1 \equiv \xi$ a small quantity and $\theta_r = (2r + 1)\pi/n$, the general term of $f(x)$, viz.

$$(1 - x \cos \theta_r)\{1 + x^2 - 2x \cos \theta_r\}^{-3/2},$$

may be written

$$\{x(1 - \cos \theta_r) - \xi\}\{2x(1 - \cos \theta_r) + \xi^2\}^{-3/2}$$

and expanded in terms of ξ, as

$$f(x) = \tfrac{1}{2}x^{-1/2}(2 - 2\cos \theta_r)^{-1/2} - \xi x^{-3/2}(2 - 2\cos \theta_r)^{-3/2}$$
$$- \tfrac{3}{4}\xi^2 x^{-3/2}(2 - 2\cos \theta_r)^{-3/2} + \tfrac{3}{2}\xi^3 x^{-5/2}(2 - 2\cos \theta_r)^{-5/2} \ldots$$

Thus if

$$\Sigma_1 = \sum_{r=0}^{n-1} \operatorname{cosec}(2r+1)\pi/2n, \quad \Sigma_3 = \sum_{r=0}^{n-1} \operatorname{cosec}^3(2r+1)\pi/2n \ldots,$$
$$. \quad . \quad . \quad (11)$$

$$f(x) = \tfrac{1}{4}x^{-1/2}\Sigma_1 - \tfrac{1}{8}\xi x^{-3/2}\Sigma_3 - \tfrac{3}{32}\xi^2 x^{-3/2}\Sigma_3 + \tfrac{3}{64}\xi^3 \Sigma_5 \ldots . \quad (12\,a)$$

When the general term of $f\left(\dfrac{1}{x}\right)$, viz.

$$x^2(x - \cos \theta_r)\{1 + x^2 - 2x \cos \theta_r\}^{-3/2},$$

is written as

$$x^2(1-\cos\theta_r+\xi)\{2x(1-\cos\theta_r)+\xi^2\}^{-3/2},$$

we find

$$f\left(\frac{1}{x}\right) = \tfrac{1}{4}x^{1/2}\Sigma_1 + \tfrac{1}{8}\xi x^{1/2}\Sigma_3 - \tfrac{3}{32}\xi^2 x^{-1/2}\Sigma_3 - \frac{3\xi^3}{64}\Sigma_5 \ldots \quad (12\,b)$$

For a vacant centre

$$N = f\left(\frac{1}{x}\right) - \tfrac{1}{2}c_n = f(x) + f\left(\frac{1}{x}\right) - c_n - \{f(x) - \tfrac{1}{2}c_n\},$$

in which the bracket vanishes as the condition determining x. Hence

$$N + c_n = \tfrac{1}{4}(x^{1/2} + x^{-1/2})\Sigma_1 + \frac{\xi\Sigma_3}{8}(x^{1/2} - x^{-3/2})$$
$$- \frac{3\xi^2\Sigma_3}{32}(x^{-1/2} + x^{-3/2}) \ldots,$$

or

$$N = c_{2n} - 2c_n + \frac{\xi^2(1-\xi)}{16}(\Sigma_3 + \Sigma_1) \ldots \quad . \quad . \quad . \quad (13\,a)$$

on using $\Sigma_1 = 2(c_{2n} - c_n)$.

The equation of condition $f(x) = \tfrac{1}{2}c_n$ by use of ξ in $(12\,a)$ becomes

$$0 = \tfrac{1}{2}(c_{2n} - 2c_n) - \frac{\xi}{8}\left(1 - \frac{3\xi}{4}\right)(\Sigma_3 + \Sigma_1) + \frac{3\xi^3\Sigma_5}{64} \ldots \quad (14\,a)$$

If this is multiplied by $\xi/2$ and added to $(13\,a)$, we have

$$N = (c_{2n} - 2c_n)\left(1 + \frac{\xi}{4}\right) - \frac{\xi^3\Sigma_3}{64} + \frac{3\xi^4\Sigma_5}{128} \ldots \quad (13\,b)$$

Also, if $(4\,a)$ is multiplied by $1 + 3\xi/4$, and $\xi^3\Sigma_3$ neglected,

$$\xi(\Sigma_3 + 4c_n - c_{2n}) = 4(c_{2n} - 2c_n) + 3\xi^3\Sigma_5/8 \ldots, \quad (14\,b)$$

where on the right hand only the main term in the value of ξ is to be applied.

The values of ξ or $x - 1$ and N are thus dependent on the sums of three series, which now require consideration *.

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* A slightly different treatment may be based on Taylor's theorem with the values

$$f(1) = \tfrac{1}{2}(c_{2n} - c_n),$$
$$-f'(1) = \tfrac{1}{8}(\Sigma_3 + \Sigma_1) = \frac{1}{\pi^3}\left[2n^3 T_3 + 3n T_2\left(\log n + \gamma + \log\frac{8}{\pi} - \frac{1}{18}\right)\right],$$
$$f''(1) = -\tfrac{3}{2}f'(1), \quad \text{and} \quad f'''(1) = 18T_5 n^5/\pi^5.$$

§ 12. If we apply the trigonometrical expansion of the cosecant *

$$\sum_{s=1}^{n-1} \operatorname{cosec} s\pi/n = \sum_{s=1}^{n-1} \left[\frac{n}{s\pi} + \sum_{r=1}^{\infty} \frac{(-1)^r 2ns}{\pi(s^2 - r^2 n^2)} \right],$$

we have

$$\frac{\pi}{n} \sum_{s=1}^{n-1} \operatorname{cosec} s\pi/n = \sum_{s=1}^{n-1} \left[\frac{1}{s} + \frac{1}{n-s} + \sum_{r=1}^{\infty} \frac{(-1)^r}{rn+s} + \sum_{r=1}^{\infty} \frac{(-1)^r}{rn+n-s} \right],$$

or $\qquad\qquad\qquad\qquad\qquad\qquad$ . . . (15 a)

$$\frac{\pi}{2n} \sum_{s=1}^{n-1} \operatorname{cosec} s\pi/n = \sum_{r=0}^{\infty} \sum_{s=1}^{n-1} \frac{(-1)^r}{rn+s}, \qquad . \quad . \quad . \quad . \quad . \quad . \quad (15 b)$$

(15 a) being a general form, (15 b) only suited to the special range of s.

The series is

$$\frac{1}{1} + \frac{1}{2} + \cdots \frac{1}{n-1} + \frac{1}{2n} - \left(\frac{1}{2n} + \frac{1}{n+1} + \cdots \frac{1}{2n-1} + \frac{1}{4n} \right)$$
$$+ \left(\frac{1}{4n} + \frac{1}{2n+1} + \cdots \frac{1}{3n-1} + \frac{1}{6n} \right) \cdots \quad (16)$$

But

$$\frac{1}{1} + \frac{1}{2} + \cdots \frac{1}{n-1} + \frac{1}{2n} = \log n + \gamma - \frac{1}{12n^2} + \frac{1}{120n^4} \cdots, \quad (17 a)$$

and the series in the first bracket of (16) is the difference between this formula as written for n and $2n$, the next bracket the difference as written for $2n$ and $3n$, and so on.

The first bracket is

$$\log \frac{2}{1} + \frac{1}{12n^2} \left(\frac{1}{1^2} - \frac{1}{2^2} \right) - \frac{1}{120n^4} \left(\frac{1}{1^4} - \frac{1}{2^4} \right) \cdots,$$

---

* I am indebted to Professor Proudman for the suggestion of this method, and a rough sketch showing the term $\log \pi/2$.

and the process applied to (16) gives

$$\frac{\pi}{2n}\sum_{s=1}^{n-1}\operatorname{cosec} s\pi/n$$
$$= \log n + \gamma - \log\frac{2\,.\,2\,.\,4\,.\,4\,\ldots}{1\,.\,3\,.\,3\,.\,5\,\ldots} - \frac{1}{6n^2}\left(\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2}\cdots\right)$$
$$+ \frac{1}{60n^4}\left(\frac{1}{1^4} - \frac{1}{2^4} + \frac{1}{3^4}\cdots\right)$$
$$= \log n + \gamma - \log\frac{\pi}{2} - \frac{\pi^2}{72n^2} + \frac{7\pi^4}{43200n^4} - \cdots,$$

and so

$$\left.\begin{array}{l} c_n = \dfrac{n}{\pi}\left\{\log n + \gamma - \log\dfrac{\pi}{2}\right\} - \dfrac{\pi}{72n} + \dfrac{7\pi^3}{43200n^3} - \cdots, \\[2ex] c_{2n} - 2c_n = \dfrac{2n}{\pi}\log_e 2 + \dfrac{\pi}{48n} - \dfrac{7\pi^3}{23040n^3}\cdots, \\[2ex] c_{2n} - c_n = \dfrac{n}{\pi}\left\{\log_e n + \gamma + \log\dfrac{8}{\pi}\right\} + \dfrac{\pi}{144n} - \dfrac{49\pi^3}{345600n^3}\cdots \end{array}\right\}(18)$$

§ 13. The method is also applicable to find the sum of cubes, but with a vast difference in the labour entailed. The cube of $(15\,a)$ has terms of types a^3, $3a^2b$, and $6abc$. In the two latter it is necessary to separate each term into partial fractions, and apply the approximate summations

$$\left.\begin{array}{l} \dfrac{1}{1^2} + \dfrac{1}{2^2} + \cdots \dfrac{1}{(n-1)^2} = S_2 - \dfrac{1}{n}\cdots, \\[2ex] \dfrac{1}{1^3} + \dfrac{1}{2^3} + \cdots \dfrac{1}{(n-1)^3} = S_3 - \dfrac{1}{2n^2}\cdots. \end{array}\right\}\quad(17\,b)$$

From these with $(17\,a)$ we have

$$\left.\begin{array}{l} \displaystyle\sum_{s=1}^{n-1}\dfrac{1}{rn+s} = \log\dfrac{r+1}{r}, \quad \displaystyle\sum_{s=1}^{n-1}\dfrac{1}{(rn+s)^2} = \dfrac{1}{nr(r+1)}, \\[3ex] \displaystyle\sum_{s=1}^{n-1}\dfrac{1}{(rn+s)^3} = \dfrac{1}{2n^2}\left\{\dfrac{1}{r^2} - \dfrac{1}{r+1^{\,2}}\right\}, \end{array}\right\}\quad(17\,c)$$

which are sufficient to give a second term of order n^{-2} compared with the main term. The separate summations are very numerous, and it may be sufficient to add to this account of the method the separate totals for the three

groups, viz.

$$\text{for type } a^3, \quad 2S_3 - \frac{S_2}{n^2},$$

$$\text{,, \quad ,, \quad } 3a^2b, \quad \frac{18S_2}{n^2}\left(\log n + \gamma - \log\frac{\pi}{2}\right),$$

$$\text{,, \quad ,, \quad } 6abc, \quad -\frac{12S_2}{n^2}\left(\log n + \gamma - \log\frac{\pi}{2}\right),$$

We have then

$$\sum_{s=1}^{n-1} \operatorname{cosec}^3 s\pi/n = \frac{2}{\pi^3}\left\{ n^3 S_3 + 3nS_2\left(\log n + \gamma - \log\frac{\pi}{2} - \frac{1}{6}\right)\right\}\ldots$$

$$\cdots \quad (19\,a)$$

The difference between this formula as written with n and $2n$ is

$$\Sigma_3 = \frac{8}{\pi^3}\left\{ 2n^3 T_3 + nT_2\left(\log n + \gamma + \log\frac{8}{\pi} - \frac{1}{6}\right)\right\}, \quad (19\,b)$$

where $T_3 = \sum\limits_{r=0}^{\infty} (2r+1)^{-3}$,   and so $7S_3 = 8T_3$;

while $3S_2 = 4T_2 = \pi^2/2$.

In the case of Σ_5 it will be enough to write the main term $64n^5 T_5/\pi^5$. This gives for $(14\,b)$ the form

$$\xi n^2\left\{ 2T_3 + 3n^{-2}T_2\left(\log_e n + \gamma - \log_e\frac{\pi}{2} - \frac{1}{18}\right)\right\}$$

$$= \pi^2 \log_e 2 + \frac{\pi^4}{96n^2} + \frac{3n^4 T_5 \xi^3}{\pi^2}\cdots,$$

and in numbers

$$\xi n^2 = 3{\cdot}2521 - 13{\cdot}1749n^{-2}\log_{10} n + 5{\cdot}0738n^{-2}\ldots,$$

$$N/n = {\cdot}44127 + {\cdot}42421n^{-2} - 1{\cdot}45342n^{-4}\log_{10} n$$
$$+ {\cdot}86256n^{-1}\ldots.$$

$$\left.\begin{array}{r}\end{array}\right\} (20)$$

Table III. has been calculated from these formulæ.

§ 14. The formulæ of § 11 may be applied when n is not small to find the changes in x and N which result from small additions to the central forces. If we describe these extra forces as a central repulsion on m_1 of amount $\epsilon^2\sigma_1/a_1^2$, and a

central attraction on m_2 of amount $\epsilon^2\sigma_2/a_2{}^2$, it will correspond to using σ_1 and σ_2 in place of the units occurring in the equations (6 b). From (14 a) and (13 a, b) we find

$$\delta x = 2x\xi\sigma_2/N \quad \text{and} \quad \delta N = -\sigma_1 + x\sigma_2. \quad (21\,a)$$

The correction for a positive centre is then got by writing $\sigma_1 = \sigma_2 = 1$, and gives

$$\delta x = 2x\xi/N \quad \text{and} \quad \delta N = \xi. \quad (21\,b)$$

The correction for μ, that is to take account of the ratio $m_2 : m_1$ instead of assuming it indefinitely small, is got by writing $\sigma_2 = -\mu N x^{-3}$, and is

$$\delta x = -2\mu\xi x^{-2}, \quad \delta N = -\mu\xi N x^{-2}. \quad (21\,c)$$

§ 15. It is proposed to deal briefly with the potential of charges distributed at equal intervals on a circle, with a view to showing the mean effect and the main fluctuation in the force on a satellite. In the plane of a single ring radius a we are concerned with a potential

$$U/\epsilon = \sum_{p=0}^{n-1} \{r^2 + a^2 - 2ra \cos (\phi + 2p\pi/n)\}^{-1/2},$$

where ϕ is the angle between the radius r of a satellite and that to the nearest unit in the ring. The Legendre coefficients must be expanded in cosines of multiples of the angle $\phi + 2p\pi/n$, and note taken of the fact that

$$\sum_{p=0}^{n-1} \cos m(\phi + 2p\pi/n)$$

vanishes unless m is a multiple of n, in which case it is $n \cos m\phi$. The mean value of the attraction depends on the term independent of ϕ, which is

$$\left. \begin{array}{l} n\left[\dfrac{1}{r} + \left(\dfrac{1}{2}\right)^2 \dfrac{a^2}{r^3} + \left(\dfrac{1\cdot 3}{2\cdot 4}\right)^2 \dfrac{a^4}{r^5} + \dots\right], \quad i.\,e. \quad \dfrac{2n}{\pi r}\,\mathrm{K}\left(\dfrac{a}{r}\right), \\[3ex] \text{or} \\[1ex] n\left[\dfrac{1}{a} + \left(\dfrac{1}{2}\right)^2 \dfrac{r^2}{a^3} + \left(\dfrac{1\cdot 3}{2\cdot 4}\right)^2 \dfrac{r^4}{a^5} + \dots\right], \quad i.\,e. \quad \dfrac{2n}{\pi a}\,\mathrm{K}\left(\dfrac{r}{a}\right), \end{array} \right\} \quad (22)$$

according as r or a is the greater. K is the first complete elliptic integral. Thus for the motion of each of s satellites an approximate value of the effect of the double ring is

shown by

$$m_2\Omega_e{}^2 r^3 = \epsilon^2\left[1 - \tfrac{1}{2}c_s + \frac{3n}{4r^2}(a_1{}^2 - a_2{}^2)\dots\right]$$
$$= \epsilon^2\left[1 - \tfrac{1}{2}c_s + \frac{3n\xi a^2}{2r^2}\dots\right] \text{ for } n \text{ large,}$$

or

$$m_2\Omega_i{}^2 r^3 = \epsilon^2\left[1 - \tfrac{1}{2}c_s - \frac{nr^3}{2}\left(\frac{1}{a_1{}^3} - \frac{1}{a_2{}^3}\right)\dots\right]$$
$$= \epsilon^2\left[1 - \tfrac{1}{2}c_s + \frac{3n\xi r^2}{2a^2}\dots\right] \quad ,, \quad ,, \quad ,,$$

$$(23)$$

according as the satellites lie outside or inside the double ring. a considerable departure of $r:a$ from 1 being presumed.

The coefficient of $\cos n\phi$ is $n\left[\alpha_n\dfrac{a^n}{r^{n+1}} + \alpha_1\alpha_{n+1}\dfrac{a^{n+2}}{r^{n+3}} + \dots\right]$, where $\alpha_n = \dfrac{1 \cdot 3 \dots (2n-1)}{2 \cdot 4 \dots 2n}$; and on this term depends the fluctuation.

For positions outside the plane we are concerned with

$$\sum_{p=0}^{n-1}\left\{r^2 + a^2 - 2ra\sqrt{1-\mu^2}\cos(\phi + 2p\pi/n)\right\}^{-1/2},$$

and it may be sufficient to write the value of the mean term

$$n\left[\frac{1}{r} - \frac{a^2}{4r^3}(3\mu^2 - 1) + \frac{3a^4}{64r^5}(35\mu^4 - 30\mu^2 + 3)\dots\right].$$

When a/r is widely different from 1, the field due to a double ring is given by

$$U_e = -\frac{n\epsilon(a_1{}^2 - a_2{}^2)}{4r^3}(3\mu^2 - 1) \quad \text{outside,}$$

and by

$$U_i = n\epsilon\left[\frac{1}{a_1} - \frac{1}{a_2} - \frac{r^2}{4}\left(\frac{1}{a_1{}^3} - \frac{1}{a_2{}^3}\right)(3\mu^2 - 1)\right] \quad \text{inside.}$$

In a rough way one may say that the cone $3\mu^2 = 1$ divides the lines of approach in which an electron would be repelled and its path show reflexion, from those in which the path is continued through the rings $R_n(0)$ with deviation.

4 A 2

TABLE I.—Vacant Centre.

n.	x.	$(x-1)n^2$.	N/n.	n.	x.	$(x-1)n^2$.	N/n.
2	1·73205	2·9282	·52452	10	1·031691	3·1691	·44537
3	1·326485	2·9384	·48104	11	1·026275	3·1793	·44467
4	1·187775	3·0035	·46475	12	1·022137	3·1877	·44424
5	1·12222	3·0555	·45676	15	1·014265	3·2091	·44355
6	1·0859	3·0924	·45225	16	1·012538	3·2100	·44290
7	1·063659	3·1193	·44935	18	1·0099278	3 2166	·44262
8	1·04907	3·1413	·44755	20	1·0080692	3·2241	·44268
9	1·038968	3·1564	·44631	30	1·003598	3·2382	·44206

TABLE II.—Positive Centre.

n.	x'.	N'.	n.	x'.	N'.
4	2·03384	2·27278	11	1·03784	4·92341
5	1·30323	2·48557	12	1·030961	5·35605
6	1·172735	2·84013	15	1·018718	6·67414
7	1·11379	3·23398	16	1·016188	7·09815
8	1·08106	3·64543	18	1·0124821	7·97730
9	1·060784	4·06661	20	1·0099163	8·85814
10	1·04729	4·49491			

Values in Tables I. and II. obtained by direct solution of the equations.

TABLE III.—Values derived from formula.

n.	$(x-1)n^2$.	N/n.	n.	$(x-1)n^2$.	N/n.
10	3·1710	·44545	36	3·2402	·44160
11	3·1807	·44473	37	3·2407	·44158
12	3·1886	·44418	38	3·2412	·44156
13	3·1953	·44375	39	3·2417	·44154
14	3·2009	·44341	40	3·2421	·44153
15	3·2058	·44314	41	3·2425	·44152
16	3·2100	·44291	42	3·2429	·44151
17	3·2136	·44273	43	3·2432	·44150
18	3·2167	·44257	44	3·2435	·44149
19	3·2195	·44244	45	3·2438	·44148
20	3·2219	·44232	46	3·2441	·44147
21	3·2241	·44223	47	3·2448	·44146
22	3·2260	·44214	48	3·2447	·44145
23	3·2278	·44207	49	3 2449	·44145
24	3·2295	·44201	50	3·2452	·44144
25	3·2308	·44195			
26	3·2320	·44190	55	3·2462	·44141
27	3·2332	·44185	60	3·2470	·44139
28	3·2342	·44181	65	3·2476	·44137
29	3·2352	·44177	70	3·2482	·44136
30	3·2361	·44174	75	3·2486	·44135
31	3·2369	·44171	80	3·2490	·44134
32	3·2377	·44168	85	3 2493	·44133
33	3·2384	·44166	90	3·2495	·44132
34	3·2390	·44164	95	3·2498	·44132
35	3·2396	·44162	100	3·2500	·44131

TABLE IV.—Some values of N_c/n for a multiple core.

$n.$	$cn/2.$	$Nc/n.$
10	3·862	·614
20	9·935	·503
30	16·834	·439
60	40·286	·329
90	66·254	·264

TABLE V.—Solutions for two axial ions.

$n.$	$y.$	x	$N.$	$n/N.$
2	4·588	1·803	·986	2·027
3	3·378	1·360	1·448	2·072
4	2·786	1·210	1·865	2·145
5	2·356	1·138	2·331	2·145
6	2·033	1·099	2·877	2·130
7	1·757	1·075	3·158	2·217
8	1·493	1·060	3·595	2·225
8*	·754	1·082	3·720	2·150
7*	·639	1·125	3·417	2·049
6*	·556	1·208	3·165	1·896
5*	·460	1·552	3·233	1·547

The figures starred belong to the closer position of axial ions.

PART II.—NATURAL OSCILLATIONS AND STABILITY.

§ 16. One main object in dealing with the natural oscillations of a system is to ascertain whether it is intrinsically stable or unstable. In the present case it is also important to ascertain the periods, with a view to comparison with spectroscopic results—electrical and optical.

For the two-ring scheme the number of variables in the oscillation problem is $6n$, of which the $2n$ referring to axial oscillation stand separate from the rest. Variables attaching to ions and electrons have very different coefficients of inertia, but enter on like terms into the forces derived from potential energy. As a consequence, the periods fall into two classes (P_1, P_2 say) specially related to ions and electrons respectively; and if $2\pi/p_1\omega$, $2\pi/p_2\omega$ are the periods, then $m_1p_1^2$ and $m_2p_2^2$ have values on the same scale dependent on n. This allows us to halve the number of the equations required to determine the separate types. Notwithstanding this reduction, the problem is very laborious, and the amount of work required for the case $n=2$ suggested the inquiry whether it would be possible to base a solution on asymptotic forms, and so general rather than individual, though restricted in application to larger values of n.

If we use only the terms of highest order in n, reduction to a standard form is in fact attainable. The validity of the reduction is well assured for oscillations of type P_2, and as here the motion of the mobile electron is primarily concerned, it is to these it is proper to look for any tendency to run away which instability would indicate.

§ 17. The results of the asymptotic treatment are that oscillations of type P_2 are real for displacements radial or axial, unreal (or the terms exponential) for tangential displacements. The axial oscillations of type P_1 are also real, but the approximations for motion in the plane are less trustworthy : they point to radial stability and some degree of tangential instability.

The individual solution for $n = 2$ gives real periods for oscillations of type P_1 close to the orbital periods, and real periods for oscillations of type P_2 with axial and radial displacements, but an exponential form appears in the tangential displacements. Thus, in respect to oscillations of type P_2, particular and general results agree in assigning instability to tangential displacements, while giving stability in other displacements.

§ 18. Apart from the analysis, simple considerations appear to be applicable when the number n is sufficiently great to give near neighbours a dominant influence. In the double ring each electron has ions as next neighbours, and in steady motion describes a small circle relative to each ; if some disturbance should accentuate the influence of one of them, the relative orbit would tend to elliptic form. This points to radial stability and tangential instability. On the other hand, in the single ring (with core) each electron has electrons as next neighbours, and if the influence of one is exaggerated the tendency is to a hyperbolic form of orbit ; which points to radial instability and tangential stability.

The second consideration is that a position of *rest* for a negative charge *in the straight line* between two positive charges is unstable. In the double ring it appears that neither the inertia of orbital motion nor the departure from alignment is adequate to overcome this tendency to instability. The argument applied to the single ring (with core) suggests tangential stability.

Again, in respect to axial motion the displacement in the oscillation equation is associated with the inverse cube of distance as coefficient. For next neighbours, therefore, we are concerned with a factor varying as n^3, while for the action of the central core there is only variation as n.

Hence in the single ring with core the repulsive action of next neighbours must be dominant for larger values of n— that is, the axial displacements unstable. The same argument suggests for the double ring a thoroughly stable position in respect to axial displacements.

§ 19. The individual solutions found for the single ring with core comprise the cases of two electrons with core $+2$ or $+3$, and three electrons with core $+3$ or $+4$. Axial displacements are here oscillatory. Radial and tangential displacements both show instability—for two electrons by the appearance of a pure exponential, for three electrons by a complex form in which the real exponential has either sign.

The asymptotic solution gives axial instability ; the approximations for plane motion are less trustworthy, but point to tangential stability and some degree of radial instability. The general conclusion for a single ring of n electrons with point-core $+ne$ is that radial instability always exists, that axial instability accompanies increase of n, while tangential instability disappears with increase of n.

§ 20. There is a marked difference in the character of the instability attaching to the two schemes. Consider first the single ring with core. With n electrons and a core $+n$, there is axial instability when n is sufficiently great. This type of instability is taken to be fatal, and the question arises whether relief can be found by distributing the electrons in several rings. To obtain a first clue a less exact method of calculation was followed, in which one electron is displaced from the plane containing other electrons and the core—that is, the mutual displacements of these elements were ignored. The condition obtained is probably less stringent than the real conditions.

On this basis it appears that up to $n=9$ a core $+n$ will give axial stability to n electrons ; but for greater values of n (number of electrons) stability demands a rapidly increasing excess of the core number. A tabulated result of calculations made by this simplified method suggests that relief can be found by distributing the electrons in a succession of rings, which need not exceed but must reach seven in dealing with the greatest value of n required. These figures may be modified by more exact treatment, *cf.* § 32. The effect of distribution in several rings on radial and tangential displacements has not been examined.

§ 21. Consider now with respect to the double ring the question of tangential instability, the only type of which

there is clear evidence in that scheme. With rings of positive and negative elements P and N, if all the N's are displaced through an angle $2\pi/n$, a new configuration of steady motion is reached. At first this instability appeared significant as providing the occasion for such a transition, but on reflexion a wider view is suggested. The instability is to be regarded as indicating that the whole motion consists in the passage from one of these configurations to others in succession.

Now, the equations set up for oscillation about the two-ring position are not adequate to deal with this finite transition—a problem of the motion of $2n$ bodies under attractive and repulsive forces in less specialized positions. The forecast I make of the motion (without calculation) is that if N is displaced tangentially towards P, there will be a gradual increase in the rate of approach, and N will be carried a little beyond P and describe a loop about it before

Fig. 4.
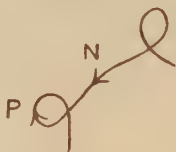
proceeding to the next special position midway between two P's. In the loop phase or near the passage through the apsidal position each N is under the almost exclusive control of the next adjacent P. As a consequence, kinetic and potential energies and acceleration are all on a greatly-increased scale in this part of the motion, much as in the apsidal phase of motion in a hyperbola or elongated ellipse. The distance between N and P is here of the order $a_1 - a_2$, say $\rho(a_1 - a_2)$ or $\rho a \xi$ at the apse. The acceleration at this apse is then $e^2/m_2 \rho^2 a^2 \xi^2$, whereas in the steady motion it is $N e^2/m_1 a^2$; thus as $N \propto n$, and $\xi \propto n^{-2}$ when n is great, the first acceleration will vary as n^4/m_2 and the second as n/m_1, a very wide disparity.

The system has been treated by mechanical rather than electromagnetic methods, since electrostatic force has been used and a constant inertia. This treatment, however, should give an approximate value of acceleration ; and since radiation varies as the square of acceleration, it would appear that in the brief space belonging to the loop phase, radiation would be on a vastly greater scale than in the rest

of the motion. Thus we have a succession of stages in each of which occurs a pulse of radiation, with no appreciable radiation between the pulses. How the interval of time between the pulses may be related to the period found in Part I. is not known.

If it were permissible to apply the formula, radiation $\propto \lambda^{-1}$, to the comparison of the above radiation for different values of n, we should get $\lambda \propto (\rho a)_n \times n^{-2}$; but the manner in which ρa may change with n is not known.

Now, in Bohr's theory there is what appears to be an arbitrary assumption of unitary stages under circumstances in which no step or stopping place occurs. The conditions suggest rather an invariable orbit if radiation is ignored, or a smooth spiral if it is taken into account. If results yielded by Bohr's theory are held to be in agreement with experimental knowledge, there still remains an absence of mechanism to explain its operation. If the above description of the whole motion in the two-ring scheme is correct in outline, a succession of stages is realized without abandoning the conception of electromagnetic radiation.

§ 22. The above account of stability is, for the two-ring system, based on oscillations of the special configuration. When the wider view of the whole motion is taken, the position in respect to tangential instability is certainly eased, but the extent of the relief is uncertain. As positive and negative elements still remain in close conjunction, the position is not worsened in respect to axial and radial stability. But it is natural to expect either an interruption or a finite modification of the oscillations at regular intervals.

A brief account will now be given of the actual periods obtained in examining the question of stability. When the atomic numbers are not small the periods for axial and radial oscillation tend to agreement. The periods (quite numerous) of type P_2 are comprised within a range represented approximately by a factor $\cdot 873$, and the shortest period has the asymptotic value $2\pi/16\cdot72n\omega$. The *range* is certainly very near to that of the extreme columns of the K series. An identification of this formula with the figures of the K(γ) series, or $2\pi/14\cdot46n\omega$ with those of the α series, involves a determination of a, the radius defining the scale of the configuration. It makes the number ma_n^3 gradually diminish with increase of n; or if A_n is atomic weight, it gives to $A_n a_n^3$ a fairly constant value $1\cdot32 \times 10^{-25}$ with some rise for lower values of n. For example, the use of the K(α)

column gives in the case of Barium ($n=56$) $a = \cdot 99 \times 10^{-9}$, $Aa^3 = 1 \cdot 33 \times 10^{-25}$.

The corresponding series of periods of type P_1 is not limited to a narrow range; its shortest periods are about 50 times those of the shortest period above (γ) or about 43 times those of the other extreme (α). The table in Sommerfeld's Atombau gives no data in which the elements of K and M series overlap, but the above factors are somewhat in excess of values I get by comparison of M figures with extrapolated values of the K series.

There is also a special ionic period, due to the axial oscillation of a central ion, of much greater duration than those enumerated above. When n is great, the factor connecting it with the shortest period above is about $32n^2/9$. The numerical value $\lambda = 3 \cdot 73 \times 10^{-5}$ then corresponds to $\cdot 388 \times 10^{-8}$ in the K(α) series for Barium in the above interpretation.

§ 23. Consider now the period of revolution of satellites. When corrections due to the rings are ignored, the motion of a single satellite is given by $m_2 \Omega^2 r^3 = \epsilon^2$. If the period $2\pi/\Omega$ is identified with Λ/V, the numerical connexion of r and Λ is given by $(r \times 10^8)^3 \times 1 \cdot 405 = (\Lambda \times 10^5)^2$. For an external satellite, Λ will be of order 10^{-5} if r is of order 10^{-8}. Thus a period in the range of light corresponds with such a value of r as appears in Chemistry in the rôle of radius of activity. Also the influence of the rings is negligible if a is of the order suggested in the last paragraph, viz. a/r is then small; but corrections can be applied by use of (23).

For an internal satellite the same equation is $(r \times 10^{10})^3 \times 1 \cdot 405 = (\Lambda \times 10^8)^2$. Identification with the L series would here make r of the order 10^{-10}, and the correction for the rings would be negligible as r/a is small.

A value of the ratio r/a near to 1 would involve a serious perturbation in the mutual action of satellite and rings, which may be the reason for the strong action of ultra-violet light.

If the axial oscillation of satellites has a period $2\pi/q\Omega$, then for $s=1$, 2, 3 I find $q=1$, $1 \cdot 155$, $1 \cdot 52$; also I find axial instability for $s=4$. For the cases of 2 and 3 satellites, $m_2 \Omega^2 r^3/\epsilon^2 = \cdot 75$ and $\cdot 423$ respectively. These cases show radial and tangential instability. Of the more complicated orbits to which this instability points, the elliptic is the more permanent form, liable, however, to a stronger perturbation from the double ring than would attach to the circular orbit.

The quasi-hyperbolic type is transient, and points to interchange with the status of free electrons.

Setting aside the cases of $R_n(+, +)$, where stability has not been examined, the most permanent forms of structure are $R_n(0)$, $R_n(+)$, and $R_n(+)$ with one satellite, two forms neutral, one with a single positive residue. Less permanent is the type with one negative residue (viz. two satellites), and in a still less degree that with a double negative residue (viz. three satellites).

§ 24. In respect to any atomic scheme there are two crucial tests to be faced—the question of stability which belongs to the domain of mathematics, and that of comparison with assured facts in the domains of Physics and Chemistry. As regards the problem of stability, examined with reference to the present scheme and that of multiple core, although the methods used are largely approximative, I believe the results to be substantial in their bearings on both schemes and in the main correct. The existence of instability in both schemes, though of differing types, demands a discrimination as to the fatal or admissible character of the instability, and in the latter case as to the function which it may be held to discharge. This again carries with it the suggestion that the activity of groups of atoms and free electrons as displayed in the world of phenomena may be dependent on some degree of instability.

The application of the second test, with the task of interpreting an abstract mathematical theory, is one that calls for a wide and intimate knowledge of experimental work, and especially of results of recent research which seem to probe the nature of the atom. To such knowledge I can lay no claim : and consequently I have felt much uncertainty in interpreting the present scheme, and some hesitation in criticizing what appear to be weak points in other theories. For this there is no remedy but an appeal to readers who may agree with my opinion that the multiple-core scheme involves an essential irrationality, may be prepared to consider the present alternative scheme, and can bring to the matter a fuller knowledge of the relevant branches of Physics.

Mathematical Theory.

§ 25. The first problem examined was that of oscillation confined to radial displacements. The limitation is realized by supposing this displacement the same for all elements lying on one ring, and also supposing all elements to have

an equal angular velocity determined by the conservation of angular momentum. The radial components then furnish equations

$$m_1\epsilon^{-2}r_1{}^2(\Omega^2 r_1 - \ddot{r}_1) = f_1(X) - \tfrac{1}{2}c_n, \quad \Big\}$$
$$m_2\epsilon^{-2}r_2{}^2(\Omega^2 r_2 - \ddot{r}_2) = f_2(X) - \tfrac{1}{2}c_n; \quad \Big\} \qquad . \quad . \quad (24)$$

where to secure symmetry the notation of (6) is altered, viz.

$$f_2(x) = f(x), \quad f_1(x) = f\left(\frac{1}{x}\right).$$

With x and 1 for original radii. ρ_1 and ρ_2 for displacements $r_1 = x + \rho_1$ and $r_2 = 1 + \rho_2$ make $X - x = \rho_1 + x\rho_2$; and the equation of angular momentum is

$$\Omega(m_1 r_1{}^2 + m_2 r_2{}^2) = \omega(m_1 x^2 + m_2),$$
or
$$\Omega = \omega(1 - 2x^{-1}\rho_1 - 2\mu x^{-2}\rho_2).$$

If then $\ddot{\rho}_1 = -q^2\omega^2\rho_1$, the left-hand members of (24) are

$$N_\mu\{1 + (q^2-1)x^{-1}\rho_1 - 4\mu x^{-2}\rho_2\}$$

and
$$\mu N_\mu x^{-3}\{1 + (q^2+3)\rho_2 - 4x^{-1}\rho_1 - 4\mu x^{-2}\rho_2\}.$$

Further, we have

$$f_1(X) - \tfrac{1}{2}c_n = f_1(x) - \tfrac{1}{2}c_n + (\rho_1 - x\rho_2)f_1{}'(x)$$
$$= N_\mu + (\rho_1 - x\rho_2)f_1{}'(x):$$

and the equations then stand

$$(\rho_1 - x\rho_2)f_1{}' = x^{-1}N_\mu\{(q^2-1)\rho_1 - 4\mu x^{-1}\rho_2\}.$$
$$(\rho_1 - x\rho_2)f_2{}' = \mu x^{-1}N_\mu\{(q^2+3)\rho_2 - 4x^{-1}\rho_1 - 4\mu x^{-2}\rho_2\}; \quad \Big\} \quad . \quad (25)$$
whereupon
$$\mu N_\mu(q^2-1)^2 + x^4 f_2{}'(q^2-1) - 4\mu x^2 f_2{}' = 0$$

represents to the first order in μ the result of elimination. The small root is $q^2 - 1 = 4\mu x^{-2}$, and the ratio of displacements associated with it is $\rho_1 = x\rho_2$ to the first order. The large root is $q^2 - 1 = -x^4 f_2{}'(x)/\mu N_\mu$, and the ratio of displacements $\mu\rho_2 f_1{}'(x) = x^2\mu_1 f_2{}'(x)$. Here ρ_1 and $\mu\rho_2$ are on the same scale of magnitude ; and, indeed, when $x-1$ is small, *i. e. n* not small, $f_1{}'$ and $f_2{}'$ are opposite in sign and nearly equal in magnitude, making $\rho_1 + \mu\rho_2 = 0$ or $m_1\rho_1 + m_2\rho_2 = 0$ a first approximation. The sign of $f_2{}'(x)$ or $f'(x)$ is negative and the oscillations real : a limiting value $q = 16 \cdot 72n$ when *n* is great, is found by using the value of $f'(1)$ given above (§ 12, footnote).

This root appears in more general equations as attaching

to sums of displacements in one ring, and provides a useful check on results obtained by approximative methods. The periods differ widely, and the fact that displacements ρ_1 and ρ_2 are of like order for one, while $m_1\rho_1$ and $m_2\rho_2$ are of like order for the other, suggests a principle which may be applied to separate the groups, and thus halve the number of equations needed to determine the periods in each group.

§ 26. To explain this method, let $x_1 x_2$ be typical coordinates respectively with positive and negative charges, and let U_2 stand for the second-order term in the potential energy. In dealing with the first group of periods we may omit inertia terms with m_2 and write $\dfrac{\partial U_2}{\partial x_2} = 0$, by use of which a new form of U_2 is reached containing only variables x_1, and accordingly the group of equations with m_1 as inertia coefficient is formed. These equations give values in which the ratio μ is neglected, and we can use x and N as tabulated in lieu of N_μ.

In the second group, μq^2 is finite when μ is accounted small, and so q is a large number; hence those inertia terms which are linear in q may be ignored in finding a first approximation. For the motion of positive charges, x_1 appears on the inertia side with a multiplier $m_1 q^2 \omega^2$, on the other side x_1 and x_2 have as a multiplier ϵ^2/a^3, which is of order $m_1 \omega^2$. Hence x_2 is of order $q^2 x_1$ or x_1 of order μx_2; and so variables x_1 may be ignored on the potential side of both sets of equations. The first set then gives x_1 in terms of x_2 with a multiplier varying as q^{-2} or as μ, and the second set determines a group of values of μq^2.

The separation presumes that we are content with the main terms in the sense that order is determined by the small number μ, and the groups present q^2 of the one and μq^2 of the other as numbers on the same scale.

§ 27. We now apply this to oscillations in the plane for $n = 2$. It is convenient to mark with an odd index coordinates referring to positive charges mass m_1, and an even index for negative charges mass m_2. If $r_1 = a_1 + \rho_1$ and $\theta_1 + \omega t$ attach to a displaced position, we can put $a_2 = a_4 = 1$ and $a_1 = a_3 = \sqrt{3}$, ignoring the μ correction to the ratio $a_1 : a_2$. The potential energy is given by

$$U/\epsilon^2 = D^{-1}(1, 3) + D^{-1}(2, 4) - D^{-1}(1, 2) - D^{-1}(1, 4) - D^{-1}(3, 2) - D^{-1}(3, 4),$$

and we are concerned with the second order terms in ρ, θ, viz. :

$$
\begin{aligned}
U_2/\epsilon^2 = \ & \frac{1}{32}\left[5(\rho_3-\rho_1)(\theta_4-\theta_2)+\sqrt{3}(\rho_4-\rho_2)(\theta_3-\theta_1)\right] \\
& + \frac{(\rho_2+\rho_4)^2}{8} + \frac{\sqrt{3}(\rho_1+\rho_3)^2}{72} - \frac{1}{32}\left[-(\rho_2{}^2+\rho_4{}^2)\right. \\
& \left. + 5(\rho_1{}^2+\rho_3{}^2)+3\sqrt{3}(\rho_1+\rho_3)(\rho_2+\rho_4)\right] \\
& + \frac{(\theta_4-\theta_2)^2}{16} + \frac{\sqrt{3}(\theta_3-\theta_1)^2}{48} - \frac{9}{64}\left[(\theta_2-\theta_1)^2\right. \\
& \left. + (\theta_4-\theta_1)^2+(\theta_3-\theta_2)^2+(\theta_4-\theta_3)^2\right].
\end{aligned} \quad (26)
$$

To deal with oscillations of type P_1 this form is reduced by use of $\dfrac{\partial U_2}{\partial \rho_2}=0\ldots$, with the result

$$
\begin{aligned}
U_2' &= -\frac{(9-\sqrt{3})\epsilon^2}{144}\left[2(\rho_1+\rho_3)^2+3(\theta_3-\theta_1)^2\right] \\
&= -\frac{m_1\omega^2}{4}\left[2(\rho_1+\rho_3)^2+3(\theta_3-\theta_1)^2\right]. \quad (27)
\end{aligned}
$$

When each displacement varies as $e^{p\omega t}$, the equations of oscillations P_1 are

$$
\left.\begin{aligned}
(p^2-1)\rho_1-2\sqrt{3}p\theta_1 &= \rho_1+\rho_3, & 3p^2\theta_1+2\sqrt{3}p\rho_1 &= -\tfrac{3}{2}(\theta_3-\theta_1) \\
(p^2-1)\rho_3-2\sqrt{3}p\theta_3 &= \rho_1+\rho_3, & 3p^2\theta_3+2\sqrt{3}p\rho_3 &= +\tfrac{3}{2}(\theta_3-\theta_1),
\end{aligned}\right\} \quad (28)
$$

leading to $p^2=0$ and $p^2=-1$, the last occurring three times. If μ were taken into account these roots would be unequal, and we find for the type P_1, stable oscillations in the plane with a period near to that of orbital revolution.

For the group P_2, coordinates of odd index are omitted from U_2, and

$$
U_2'' = \frac{\epsilon^2}{32}\left[5\rho_2{}^2+5\rho_1{}^2+8\rho_2\rho_4-7\theta_2{}^2-7\theta_1{}^2-4\theta_2\theta_4\right]. \quad (29)
$$

The equations of oscillation are then

$$
m_2p^2\omega^2\rho_2 = -\frac{\epsilon^2}{16}(5\rho_2+4\rho_4), \quad m_2p^2\omega^2\theta_2 = \frac{\epsilon^2}{16}(7\theta_2+2\theta_4),
$$

$$
m_2p^2\omega^2\rho_4 = -\frac{\epsilon^2}{16}(5\rho_4+4\rho_2), \quad m_2p^2\omega^2\theta_4 = \frac{\epsilon^2}{16}(7\theta_4+2\theta_4),
$$

or

$$
\left.\begin{aligned}
m_2 p^2 \omega^2 (\rho_2 + \rho_4) &= -\frac{9\epsilon^2}{16}(\rho_2 + \rho_4), \\
m_2 p^2 \omega^2 (\rho_2 - \rho_4) &= -\frac{\epsilon^2}{16}(\rho_2 - \rho_4), \\
m_2 p^2 \omega^2 (\theta_2 + \theta_4) &= +\frac{9\epsilon^2}{16}(\theta_2 + \theta_4), \\
m_2 p^2 \omega^2 (\theta_2 - \theta_4) &= \frac{5\epsilon^2}{16}(\theta_2 - \theta_4).
\end{aligned}\right\} \quad . \quad . \quad (30)
$$

Terms linear in p having been omitted on the inertia side, the above gives main terms in the value of p^2. The first of the real periods, numerical value $p = 71 \cdot 2 \sqrt{-1}$, will be found to agree with the result of § 25 for purely radial oscillation.

For θ_2 and θ_4 the values are exponential, and as the exponent may have either sign, there is a clear case of instability in respect to oscillations of tangential character in special connexion with the motion of electrons.

For oscillations in a direction z perpendicular to the plane of the orbit, we have

$$
\left.\begin{aligned}
U_2/\epsilon^2 &= -\frac{(z_1 - z_3)^2}{16a_1^3} - \frac{(z_2 - z_4)^2}{16a_2^3} + \frac{1}{2(a_1^2 + a_2^2)^{3/2}} \times \\
&\quad [(z_1 - z_2)^2 + (z_1 - z_4)^2 + (z_3 - z_2)^2 + (z_3 - z_4)^2], \\
&= -\frac{(z_1 - z_3)^2}{48\sqrt{3}} - \frac{(z_2 - z_4)^2}{16} \\
&\quad + \tfrac{1}{16}[(z_1 - z_2)^2 + (z_1 - z_4)^2 + (z_3 - z_2)^2 + (z_3 - z_4)^2].
\end{aligned}\right\} \quad (31)
$$

The order of equations being only half that for motion in the plane, we may dispense with the special method; the types are then given by

$$
\left.\begin{aligned}
m_1 \ddot{z}_1 &= \frac{\epsilon^2(z_1 - z_3)}{24\sqrt{3}} - \frac{\epsilon^2(2z_1 - z_2 - z_4)}{8}, \\
m_2 \ddot{z}_2 &= \frac{\epsilon^2(z_2 - z_1)}{8} - \frac{\epsilon^2(2z_2 - z_1 - z_3)}{8}.
\end{aligned}\right\} \quad . \quad . \quad (32)
$$

Subtracting forms for z_1 and z_3,

$$
m_1 p^2 \omega^2 = \epsilon^2 \left(\frac{1}{12\sqrt{3}} - \frac{1}{4} \right) = -m_1 \omega^2,
$$

that is $p^2 = -1$; and subtracting forms for z_2 and z_4, we have $p^2 = 0$. Addition leads to

$$\{p^2 + \tfrac{3}{26}(9 + \sqrt{3})\}(z_1 + z_3) = \tfrac{3}{26}(9 + \sqrt{3})(z_2 + z_4),$$

and

$$\{\mu p^2 + \tfrac{3}{26}(9 + \sqrt{3})\}(z_2 + z_4) = \tfrac{3}{26}(9 + \sqrt{3})(z_1 + z_3);$$

which give

$$p^2 = 0 \quad \text{or} \quad \mu p^2 = -\tfrac{3}{26}(9 + \sqrt{3}),$$

that is

$$p = \pm 47 \cdot 38 \sqrt{-1}.$$

Axial displacement therefore gives rise to true oscillations for both types P_1 and P_2.

§ 28. The discovery of tangential instability prompted inquiry as to the position in the system of multiple core. It is convenient to use coordinates relative to the core. In general, for a group comprising a mass m_0 and other masses m_r, the relative kinetic energy is given by

$$2\mathrm{T}_r = \sum_r m_r \dot{x}_r^2 - (\Sigma m_r \dot{x}_r)^2/(m_0 + \Sigma m_r), \quad . \quad . \quad (33)$$

when coordinates relative to m_0 are used. In the present case the omission of the last term involves a modification of periods of the order μ. In seeking the main term in the problem of two electrons and a core $+2\epsilon$, we may therefore write

$$2\mathrm{T}_r = m_2(\dot{x}^2 + \dot{y}^2 + \dot{x}'^2 + \dot{y}'^2),$$

and pass to polar coordinates $1 + \rho$ and $\theta + \omega t$.

Since $\quad \mathrm{U}_2/\epsilon^2 = \tfrac{1}{8}(\rho + \rho')^2 + \tfrac{1}{16}(\theta - \theta')^2 - 2\rho^2 - 2\rho'^2,$

the equations of oscillation, taking account of $m_2\omega^2 = 7\epsilon^2/4$, are

$$\left.\begin{aligned}
(p^2 - 1)\rho - 2p\theta &= \tfrac{16}{7}\rho - \tfrac{1}{7}(\rho + \rho'), \\
(p^2 - 1)\rho' - 2p\theta' &= \tfrac{16}{7}\rho' - \tfrac{1}{7}(\rho + \rho'), \\
p^2\theta + 2p\rho &= -\tfrac{1}{14}(\theta - \theta'), \\
p^2\theta' + 2p\rho' &= \tfrac{1}{14}(\theta - \theta'),
\end{aligned}\right\} \quad . \quad . \quad (34)$$

which give

$$(p^2 - \tfrac{23}{7})(\rho - \rho') = 2p(\theta - \theta'), \quad (p^2 + \tfrac{1}{7})(\theta - \theta') = -2p(\rho - \rho')$$

and

$$(p^2 - 3)(\rho + \rho') = 2p(\theta + \theta'), \quad p^2(\theta + \theta') = -2p(\rho + \rho').$$

The second pair gives $p^2 = 0$ or $p^2 + 1 = 0$. The first pair gives

$$p^4 + 6p^2/7 - 23/49 = 0,$$

$$p^2 = (-3 \pm 4\sqrt{2})/7 = \cdot 3795 \quad \text{or} \quad -1 \cdot 2367,$$

the first root implying instability which attaches to radial and tangential coordinates. Axial oscillation is here stable. With core $+3\epsilon$ the quadratic is altered to

$$p^4 + 10p^2/11 - 35,121 = 0.$$

It is not proposed to write out the work for three electrons with cores $+2\epsilon$, $+3\epsilon$, $+4\epsilon$. The final equation is a quartic of simple character.

By way of exploration of the source of instability, these problems were also solved with the single modification of repulsion to attraction in the mutual action of electrons, masses and intensities of force as before, with the result that complete stability was found.

§ 29. A search for further information in respect to $R_n(0)$ may be pursued in two ways—either by examination of individual cases as $n = 4$, 6, 8..., or by use of asymptotic formulæ. The tedious work which the former course would entail seemed prohibitive, and in respect to the course actually taken it must be understood that approximations are based on the treatment of $1/n$ as a small quantity. The treatment of axial displacement is simple, and yields

$$U_2/\epsilon^2 = \sum_r (z - z_r')^2/2D_r^3 - \sum_s (z - z_s)^2/2D_s^3 \quad . \quad (35\,a)$$

as expression for the terms containing any one z, D_s then being distance between this element and any repulsive element, or $D_s = 2a \sin s\pi/n$; and D_r distance between the same element and any attractive element, or approximately $D_r = 2a \sin(2r + 1)\pi/2n$.

The second order terms in potential energy which contain displacements in the plane are more complicated, and here approximation is needed. For two points whose distance in steady motion is given by $D^2 = a^2 + a'^2 - 2aa' \cos\psi$, and for which the increments of coordinates are (ρ, θ), (ρ', θ'), we find the terms of second order in D^{-1} to be

$$\left.\begin{aligned}
&-(\rho^2 + \rho'^2 - 2\rho\rho'\cos\psi)/2D^3 + 3\{a\rho + a'\rho' \\
&\qquad\qquad - (a\rho' + a'\rho)\cos\psi\}^2/2D^5, \\
&+ (\theta - \theta')\sin\psi[a\rho' + a'\rho - 3aa'\{a\rho + a'\rho' \\
&\qquad\qquad - (a\rho' + a'\rho)\cos\psi\}/D^2]/D^3, \\
&- aa'(\theta - \theta')^2[\cos\psi - 3aa'\sin\psi/D^2]/2D^3.
\end{aligned}\right\} \quad (36)$$

In the case of attraction the terms containing (ρ, θ) are got by writing $(2r+1)\pi/n$ for ψ, (ρ_r', θ_r') for (ρ', θ') and summing for values of r from 0 to $n-1$, when such a sum gives the terms in $-U_2/\epsilon^2$ which involve (ρ, θ). In dealing with repulsion $a=a'$, $2s\pi/n$ takes the place of ψ, (ρ_s, θ_s) replace (ρ', θ'), and s ranges from 1 to $n-1$, when the sum gives the terms in U_2/ϵ^2 which contain (ρ, θ).

This exact expression is simplified when n is great, for $a \sim a'$ is of order n^{-2} and D of order n^{-1} for the near neighbours whose action is most influential.

The terms of highest order which contain a given (ρ, θ) are then

$$U_2/\epsilon^2 = \sum_r \{ (\rho-\rho_r')^2/2 - a^2(\theta-\theta_r')^2 \}/D_r^3$$
$$+ \sum_s \{ -(\rho-\rho_s)^2/2 + a^2(\theta-\theta_s)^2 \}/D_s^3 \quad . \quad (35\,b)$$

Thus a form of the same type as $(35\,a)$ is realized, but only in virtue of the approximations used.

On the inertia side also a greater simplicity attaches to the coordinate z than to ρ, θ; for if each has a factor $e^{\rho\omega t}$, p^2z occurs in the axial motion, $(p^2-1)\rho - 2p a\theta$ and $p^2 a\theta + 2p\rho$ in the plane motion. No sufficient simplification is attainable unless the terms linear in p can be neglected, and this requires p to be great. Now p^2 varies as n^3/N and ultimately as n^2, *i. e.* its main term has a form $n^2\chi$ where χ varies from one root to another. The omission of linear terms is therefore permissible for large values of n so long as χ does not become small. This case of difficulty does not occur in dealing with the type P_2, but it appears in respect to type P_1 and also in the work for a single ring with core. Oscillations P_2 have the additional advantage of showing an extra factor m_1/m_2 in the value of p^2 which greatly improves the approximation.

§ 30. Axial oscillations of type P_2 are represented by an equation

$$m_2 p^2\omega^2 z = -\epsilon^2 \{ \sum_r (z-z_r')/D_r^3 - \sum_s (z-z_s)/D_s^3 \}$$

or

$$\mu p^2 z = -\frac{1}{8N} \{ \sum_r (z-z_r') \operatorname{cosec}^3 (2r+1)\pi/n$$
$$-\sum_s (z-z_s) \operatorname{cosec}^3 s\pi/n \}. \quad (37)$$

In accordance with § 26 we may omit z_r', and the sum of variables then shows a period given by

$$\mu p^2 = -\Sigma_3/8N = -n^2 T_3/\pi^3 \log 2 \quad \text{or} \quad p^2 \doteq -(16\cdot72n)^2$$

in agreement with § 25. For other periods the coefficients

of $z - z_s$ enter into the calculations. These coefficients are equal for s and $n - s$, and for near neighbours vary inversely as the cube of the smaller of these numbers. We get an approximate account of the mutual action if we take only next neighbours for which $s = 1$. For example, the equation for z_2 with next neighbours z_1 and z_3 stands

$$\left. \begin{array}{c} \mu p^2 z_2 + \alpha z_2 - \beta(z_3 - z_1 + z_2 - z_3) = 0 \\ \text{or} \quad (\mu p^2 + \alpha - 2\beta)z_2 + \beta(z_1 + z_3) = 0, \\ \text{or say} \qquad\qquad y z_2 + z_1 + z_3 = 0, \end{array} \right\} \qquad . \quad (38)$$

in which

$$\alpha = \Sigma_3/8N = 2n^3 T_3/\pi^3 N, \quad \beta = n^3/8\pi^3 N. \quad . \quad (39)$$

Equations (38) written for each variable constitute a cyclic group, which may be treated by the method of determinants or as an equation of finite differences. The latter method gives $z_n = (-1)^n (A \sin n\gamma + B \cos n\gamma)$ where $y = 2 \cos \gamma$. The cyclic character is then expressed by the conditions $z_{n+1} = z_1$ and $z_{n-2} = z_2$, which for n even require $\sin n\gamma/2 = 0$ or $\sin \gamma = 0$, for n odd $\cos n\gamma/2 = 0$ or $\sin \gamma = 0$. The determinant * itself is given by

$$\tfrac{1}{2}\Delta_n = \cos n\gamma + (-1)^{n+1}. \quad . \quad . \quad . \quad (40)$$

For n even the solutions are given by
$$n\gamma = \pi(2, 4, \dots \ 2n) ;$$
for n odd the solutions are given by
$$n\gamma = \pi(1, 3, \dots \ 2n - 1).$$
$$\left. \begin{array}{c} \\ \\ \\ \\ \end{array} \right\} \quad . \quad . \quad . \quad (41)$$

The value $y = 2$ or $\gamma = 2\pi$ occurs only with n even; $y = -2$ or $\gamma = \pi$ is common to the two series and involves $z_1 = z_2 = z_3 = \dots$, which was assumed as basis of the problem in § 25. All other roots occur in pairs which give equal values to y or $2 \cos \gamma$; but these equal roots of the approximate equation will no doubt be replaced by closely adjacent roots with more exact treatment. It is convenient to refer to the values $y = -2$ and $+2$ as extreme values, though the latter only occurs with n even, this extreme for n odd being $2 \cos \pi/n$.

$$* \ \Delta_4 = \begin{vmatrix} y, & 1, & 0, & 1 \\ 1, & y, & 1, & 0 \\ 0, & 1, & y, & 1 \\ 1, & 0, & 1, & y \end{vmatrix}$$

It would be of interest to get solutions of the determinant with two other neighbours, in which the first row of Δ_6' would be $y, 1, k, 0, k, 1$; where we could suppose $k < 1$.

To the value $y = -2$ corresponds in (38)

$$\mu p^2 + \alpha - 2\beta = -2\beta \quad \text{or} \quad \mu p^2 + \alpha = 0,$$

the root attaching to the sum of variables. The other extreme gives $\mu p^2 = -\alpha + 4\beta$, a negative number since

$$\alpha : 4\beta = 4T_3 = 4\cdot2072 ;$$

this corresponds to a period $2\pi/14\cdot6n\omega$, the number $16\cdot72$ being reduced by a factor $\cdot873$ or $\sqrt{3\cdot2072/4\cdot2072}$.

The reduced equations for ρ agree with those for z, *cf.* (35, *a, b*) ; in view of these reductions we find that the periods for radial tend to differ from those of axial displacements by amounts which diminish as n is increased. For tangential displacements it appears from (35 *b*) that we obtain exponentials with exponents $\sqrt{2}$ times the values found in the oscillations.

§ 31. For oscillations of type P_1 we have to deal with variables for both rings, and revert to the plan of separation by odd and even indices. If we retain only the first term in the repulsive series and the first in the attractive, then for axial motion a specimen equation is

$$m_1 p^2 \omega^2 z_3 = -\frac{\epsilon^2 n^3}{\pi^3 a^3} \{2z_3 - z_2 - z_4 - \tfrac{1}{8}(2z_3 - z_1 - z_5)\ldots\}$$

or $\quad p^2 z_3 = -\dfrac{n^3}{N\pi^3} \{2z_3 - z_2 - z_4 - \tfrac{1}{8}(2z_3 - z_1 - z_5)\ldots\} \quad . \quad . \quad (42)$

In accordance with § 26, we write

$$\frac{\partial U_2}{\partial z_2} = 0 = \frac{\partial U_2}{\partial z_4} \ldots,$$

then taking only the first terms of these, viz. $2z_2 - z_1 - z_3 = 0$, $2z_1 - z_3 - z_5 = 0$, we can clear (42) of variables with even index and obtain

$$\left(\frac{8N p^2 \pi^3}{3n^3} + 2\right) z_3 - z_1 - z_5 = 0. \quad . \quad . \quad (43)$$

As $z_{1, 3, 5}$ refer to consecutive ions, the method used above is applicable and gives to $8N p^2 \pi^3/3n^3 + 2$ a series of values ranging from -2 to $+2$, or to $N p^2 \pi^3/n^3$ values from $-3/2$ to 0. The greatest numerical value is $p/n = \cdot331 \sqrt{-1}$, or wave-length $50\cdot5$ times the least wave-length in P_2. But in treating P_2 all terms $z - z_r'$ were taken into account, here only the first ; and a fairer factor of comparison is $49\cdot2$, got by omission of factor T_3 in the first result.

For radial displacement the limitation imposed by neglecting

terms linear in p must be remembered in respect to the smaller values of p corresponding to $y+2$ small; and even at the other extreme the approximation is sensibly less trustworthy because the values of p are only $1/50$ of those for the type P_2.

§ 32. For a central core $+n\epsilon$ and a ring of n electrons the equations of oscillation are

$$m_2 p^2 \omega^2 z = \epsilon^2 \left[\sum_s \frac{z-z_s}{D_s^3} - \frac{nz}{a^3} \right], \quad m_2 p^2 \omega^2 \rho = \epsilon^2 \left[\sum_s \frac{\rho-\rho_s}{D_s^3} + \frac{n\rho}{a^3} \right],$$

$$m_2 p^2 \omega^2 a\theta = -\epsilon^2 \sum_s \frac{2a(\theta-\theta_s)}{D_s^3}, \quad . \quad . \quad . \quad (44)$$

the two latter being simplified as to the right-hand member by supposing n great, and as to the left-hand by supposing p great. A specimen equation for axial motion when only next neighbours are taken into account is

$$\left(\frac{8\pi^3 N_c p^2}{n^3} + \frac{8\pi^3}{n^2} - 2 \right) z_2 + z_1 + z_3 = 0,$$

which gives to $\dfrac{8\pi^3 N_c p^2}{n^3} + \dfrac{8\pi^3}{n^2}$ a range of values from 0 to 4. Thus for higher values of n, p^2 is necessarily positive for some part of the range with a transition taking place when $2\pi^3 = n^2$. This gives to n a value just under 8—a value which hardly justifies the approximation, and so leaves the point of transition uncertain.

For the radial displacement, $\dfrac{8\pi^3 N_c p^2}{n^3} - \dfrac{8\pi^3}{n^2}$ ranges from 0 to 4, and all values of p^2 are positive. In the tangential displacement the form taken by the sequence equation is

$$\left(\frac{4\pi^3 N_c p^2}{n^3} + 2 \right) \theta_2 - \theta_1 - \theta_3 = 0,$$

and makes $\dfrac{4\pi^3 N_c p^2}{n^3}$ range from -4 to 0.

Lastly, if the method of next neighbour is applied to the case of n electrons in one ring with a different core number, say m_c, then $\dfrac{8\pi^3 N' p^2}{n^3} + \dfrac{8\pi^3 m_c}{n^3}$ has values from 0 to 4, N' being $m_c - \frac{1}{2}c_n$. Complete axial stability then demands $2\pi^3 m_c > n^3$. This points to a much more rapid increase of m_c with increase of n than the special method referred to above (§ 20). But the oscillation method has various features of approximation,

and the real conditions probably lie between the limits suggested by the two results; and in particular the case $m_c = n = 8$ probably lies within the margin of axial stability.

§ 33. In treating oscillations for the double ring the centre was taken to be vacant, and for n large it is clear that the effect of a unit centre in altering periods of oscillation is slight. But a new period is necessarily introduced—that of the central ion itself, which may be treated in conjunction with the question of stability for central ion or electron under axial displacement: a fundamental question to which only a preliminary answer was given in Part I.

A reference to $(35\,a)$ shows that it is possible by summation of equations of motion to isolate the two sums of z coordinates for ions and for electrons. The periods thus given (and one of them is that of the central charge) are such as would follow from using the same coordinates z_1 and z_2 for each element of the separate rings, and this method is more convenient for the purpose. Thus for a central ion with z_0 as axial displacement, z_1 and z_2 coordinates *relative* to the ion, equations of motion are :—

$$\left. \begin{array}{l} m_1\ddot{z}_0 = Z - n\alpha z_1 + n\beta z_2, \\ m_1(\ddot{z}_0 + \ddot{z}_1) = Z + \alpha z_1 - \gamma(z_1 - z_2), \\ m_2(\ddot{z}_0 + \ddot{z}_2) = -Z - \beta z_2 + \gamma(z_1 - z_2); \end{array} \right\} \quad . \quad . \quad (45)$$

in which

$$\left. \begin{array}{l} \alpha = \epsilon^2/a_1{}^3, \quad \beta = \epsilon^2/a_2{}^3 = x^3\alpha, \quad \gamma = 2\beta\phi = 2x^3\alpha\phi, \\ 2\phi = \sum_{r=0}^{n-1} \{1 + x^2 - 2x\cos(2r+1)\pi/n\}^{-3/2}, \end{array} \right\} \quad (46)$$

and for n great $\quad \phi = n^3 T_3/\pi^3$ approximately.

The external force is required in a later application. Eliminating z_0, we have

$$\left. \begin{array}{l} m_1\ddot{z}_1 = (n+1)\alpha z_1 - n\beta z_2 - \gamma(z_1 - z_2), \\ m_2\ddot{z}_2 = -\beta z_2 + \gamma(z_1 - z_2), \\ \ddot{z} = -q^2\omega^2 z \quad \text{and} \quad m_1\omega^2 a_1{}^3 = N\epsilon^2, \\ (q^2 N + n + 1 - 2x^3\phi)z_1 = (n - 2\phi)x^3 z_2, \\ (\mu q^2 N - x^3 - 2x^3\phi)z_2 = -2x^3\phi z_1, \end{array} \right\} \quad . \quad . \quad (47)$$

or with

leading to

$$\mu q^4 N^2 - q^2 N x^3(2\phi + 1) + (n+1)x^3\{2\phi(x^3 - 1) - 1\} = 0, \quad (48\,a)$$

where the only approximation used is an omission of μ, and N is the value proper to the case with centre. Where a

positive centre is possible, that is for $n > 4$, the last bracket in (48 a) is positive, and the two values of q^2 real and positive. If the oscillation can be identified, this is a means of determining μ, which can be used with any value of $n \geq 4$ by calculation of ϕ.

The asymptotic value is $m_1/m_2 = 143 \cdot 8 q^2/n^4 q'^2$; the roots are $N q^2 = 2\phi/\mu = 2 n^3 T_3/\mu \pi^3$ and $N q'^2 = 3n\xi$ in asymptotic form, or with our previous value of μ, $q = 16 \cdot 72 n$ and $q' = 4 \cdot 712/n$. The number q' corresponds to the new ionic period, and q is not altered, to this order in n, from the value previously found for the sum of displacements.

For a negative unit at the centre the equations are changed

to $\quad m_2 \ddot{z}_0 = n \alpha z_1 - n \beta z_2, \quad m_1(\ddot{z}_0 + \ddot{z}_1) = -\alpha z_1 - \gamma(z_1 - z_2),$
$$m_2(\ddot{z}_0 + \ddot{z}_2) = \beta z_2 + \gamma(z_1 - z_2) ;$$

and elimination as carried out above leads to

$$\mu^2 q^4 N^2 - \mu q^2 N \{ 2x^3 \phi + n - (n+1)x^3 \} \\ - n x^3 \{ 2\phi(x^3 - 1) + 1 \} = 0. \quad (48\,b)$$

One of the roots is necessarily negative when $x > 1$, and on examination it appears that this is also true for $n = 2, 3,$ or 4, cases for which $x < 1$, that is $2\phi(x^3 - 1) + 1$ is a positive quantity. By the criterion of axial stability, therefore, the admission of a positive unit centre and the rejection of a negative unit are justified.

§ 34. The equations (45) may also be applied to test the cohesion of the system under external force in the direction of the axis. Stability is a part of such a test, but we may take a further step and inquire into the extent of internal displacements when the various elements move with a common acceleration f. Thus we write $\ddot{z}_0 = f$, $\ddot{z}_1 = 0 = \ddot{z}_2$, and examine the magnitude of $z_1 z_2$ for a given external force Z.

Multiplying the second and third by n and adding all equations, we get

$$Z = \{ (n+1)m_1 + m_2 \} f, \text{ or } Z - m_1 f = nMf,$$
$$Z + m_2 f = (n+1)Mf \text{ where } M = m_1 + m_2.$$

The three consistent equations are then

$$Mf = \alpha z_1 - \beta z_2, \quad nMf = \gamma(z_1 - z_2) - \alpha z_1,$$
$$(n+1)Mf = \gamma(z_1 - z_2) - \beta z_2,$$

of which the solution is $\qquad\qquad\qquad\qquad\qquad\qquad$. (49)

$$\{ \gamma(\alpha - \beta) + \alpha\beta \} z_1 = (\gamma - n\beta)Mf,$$
$$\{ \gamma(\alpha - \beta) + \alpha\beta \} z_2 = \{ \gamma - (n+1)\alpha \} Mf.$$

This gives

$$\{\gamma(\beta - \alpha) - \alpha\beta\}(z_2 - z_1) = \{(n+1)\alpha - n\beta\}\, \mathrm{M}\, f = \mathrm{M} f\alpha$$

approximately for n great, and so

$$\mathrm{M} f = \gamma(x^3 - 1)(z_2 - z_1) = 3\xi\gamma(z_2 - z_1),$$

or $$Z = 3n\xi\gamma(z_2 - z_1).$$

If now we write Z which is the force on one ion as ϵ^2/d^2, with the action of one atom on another at interatomic distance in view, and apply numerical values, we get

$$\frac{z_2 - z_1}{a} = \frac{1 \cdot 512}{n^2} \times \frac{a^2}{d^2};$$

and a comparison with $\dfrac{a_1 - a_2}{a} = \dfrac{3 \cdot 25}{n^2}$ shows that the *relative* displacement of the rings is small compared with the difference of radii. The value of z_1 or z_2 is much greater than their difference, viz. $\mathrm{M} f = -(\beta - \alpha)z_1$ or $Z = -3n\xi z_2$, with a numerical result $z_1/a = -na^2, 9 \cdot 75 d^2$. On this feature that the displacement of the central ion relative to either ring is on a much greater scale than their mutual displacement, is based the remark on electrolytic conduction in § 5, Part I.

If Z is not a constant but a periodic force of period $2\pi/\mathrm{Q}\omega$, the method used above will be found to give

$$\left.\begin{array}{l} \epsilon^2 a^{-3} z_1 [\mu\mathrm{Q}^4\mathrm{N}^2 - \mathrm{Q}^2\mathrm{N} x^3(2\phi + 1) \\ \qquad + (n+1)x^3\{2\phi(x^3 - 1) + 1\}] = -x^3(2\phi - n)Z, \\ \qquad\text{or also} \\ m_2 z_1 \omega^2 \mathrm{N}(\mathrm{Q}^2 - q^2)(\mathrm{Q}^2 - q'^2) = -x^3(2\phi - n)Z, \\ m_2 z_2 \omega^2 \mathrm{N}(\mathrm{Q}^2 - q^2)(\mathrm{Q}^2 - q'^2) = -(2x^3\phi - n - 1 - \mathrm{Q}^2\mathrm{N})Z \end{array}\right\} \quad (50)$$

with q and q' as in § 33. When there is no central charge the formulæ are

$$m_2 z_2 \omega^2 (\mathrm{Q}^2 - q^2) = Z_1 \quad \text{and} \quad m_1 z_1 + m_2 z_2 = 0.$$

The method could no doubt be applied to examine the scale of displacement of satellites, but not without a sensible complication of the equations.

In bringing this long task to a close I wish to acknowledge the kindness of Sir Joseph Larmor in reading earlier sketches of this paper, and making various comments, criticisms and

suggestions, which have proved of great service in the revision of the manuscript. The interest he took in this question of the freely equilibrated nucleus and other dynamical problems presented in the paper, provided a valuable stimulus in the course of work which of necessity comprised much tedious calculation. For this assistance I am grateful, and feel all the more indebted since it involved some encroachment on a very scanty leisure.

CI. *Selective Reflexion of* λ 2536 *by Mercury Vapour. By* R. W Wood [*].

IN earlier papers it was shown that there appear to be two types of selective reflexion of radiation which is very nearly in synchronism with the free period of the mercury molecule at the 2536 absorption line. One type is due to the abnormally low value of the refractive index of the vapour on the short wave-length side of the line. The change of refractive index at the boundary quartz–Hg vapour is greater than the change for a boundary quartz-vacuum, since the refractive index is less than unity, consequently we have strong reflexion for radiations immediately adjacent to the absorption line on the short wave-length side. The high value of the refractive index of the vapour for radiations of wave-length slightly greater than that of the absorption line, makes the change of index at the boundary small, consequently the reflexion for these radiations is very feeble. This was shown by reflecting the light of λ = 2536 from a quartz arc operated at high temperature (2536 broad and strongly reversed) from the inner surface of a flat prismatic quartz plate which was sealed to a quartz bulb containing mercury vapour at a pressure of several atmospheres. The reflected light was photographed with a quartz spectrograph, and only the short wave-length half of the 2536 reversed line was found on the plate.

The experiment was also tried using the 2536 line from the water-cooled quartz arc. In this case the reflecting power of the quartz-mercury vapour surface was about four times as great as the normal reflecting power of quartz in this region of the spectrum. Since in this case the light is highly homogeneous, it was inferred that the high reflectivity was the result of the absorbing power of the vapour, the case being analogous to the selective reflexion of the aniline dyes.

It occurred to me recently, however, that it would be desirable to repeat the experiment using still more homogeneous light, as a considerable portion of the 2536

radiation, even in this case, can pass through a layer of mercury vapour at room temperature, 10 cm. in thickness. We may thus interpret the result of the experiment as due to the selective reflexion of a certain very narrow range of wave-lengths in the 2536 line, just as in the previous case where the high temperature quartz mercury arc was used. I have accordingly repeated the experiment using a mercury resonance lamp at room temperature as a source of light.

As I have shown in previous papers, the radiation in this case is almost completely stopped by a layer of mercury vapour at room temperature a few millimetres in thickness.

The thick-walled bulb of fused quartz closed at one end by a prismatic plate of the same substance, which was used in the earlier work, was mounted in an electric furnace in close proximity to a thermo-couple.

The resonance lamp was mounted in such a position that its image, reflected from the inner surface of the prismatic plate, was received by the lens of the quartz camera. This adjustment was facilitated by attaching a small square of white paper to the surface of the resonance lamp, in coincidence with the area which radiated the 2536 monochromatic light, when the lamp was illuminated by the concentrated beam of 2536 light from a quartz monochromator, the light coming originally from a water-cooled quartz mercury arc.

The paper square was illuminated by a concentrated beam of white light, and by carefully adjusting the bulb in the furnace, the image of the paper, reflected from the inner surface of the plate, was seen in the camera. The paper square was then removed, and two exposures made, with the resonance lamp in operation, one with the bulb cold, the other with the bulb at 400°, the photographic plate being moved between the exposures. Just before each exposure the prismatic plate of the bulb was super-heated, by brushing it with a small pointed gas flame. This removed any condensed droplets of mercury, which sometimes formed on the inner surface of the plate.

The image reflected from the plate was much denser in the case of the exposure with the bulb hot. A number of exposures were now made giving longer times for the cases in which the bulb was cold. These showed that the reflecting power of the plate when backed by dense mercury vapour (density corresponding to 400°) was between 3·5 and 4 times as great as the normal reflecting power of quartz for the wave-length in question, a result which is in agreement with the value found in the earlier work.

CII. *Polarized Resonance Radiation of Mercury Vapour. By* R. W. Wood [*].

[Plate IX.]

IN my earlier papers on this subject I expressed the opinion that the resonance radiation of mercury vapour showed no traces of polarization, a somewhat surprising circumstance, in view of the fact that the resonance spectra of sodium and iodine are strongly polarized.

Lord Rayleigh published a short letter in 'Nature' several years ago stating that strong polarization could be observed in that portion of the excited vapour at some distance from the window through which the stimulating radiation entered, the percentage of polarization falling off as the window was approached. This indicated that the polarization was produced only by radiation not quite in synchronism with the molecule, of which we have a very marked example in the case of the light scattered by air and other gases, in which the wave-length of the radiation is very far removed from that of the absorption bands of the gas.

During the past winter I have made a further examination of the subject, and have found that the radiation is strongly polarized, but that the percentage of polarization does not appear to depend upon the distance to which the exciting radiation has penetrated. The mercury vapour was contained in a highly exhausted quartz tube with windows of the same substance ground and polished. The exciting radiation was furnished by a quartz mercury arc, the lower half of the tube being covered with cotton, over which a stream of water flowed continuously, this arrangement being necessary to prevent the reversal of the 2536 line which excites the vapour. The radiation from a wide slit, placed close to the arc, was passed in turn through a large quartz lens, a quartz prism of about 40°, and a second lens : the prism was cut parallel to the optic axis of the quartz, consequently two polarized spectra were obtained, and a polarized monochromatic radiation of wave-length 2536 could be obtained from a second slit suitably located. The plate in which this slit was cut was coated with barium platino-cyanide to facilitate its adjustment in the ultra-violet regions

* Communicated by the Author.

of the two spectra.   Figure 1 shows the arrangement of the apparatus.

A is a quartz wedge of small angle cut parallel to the axis. With monochromatic polarized light properly oriented, this shows when viewed through an analyser a fringe system, the visibility of which increases with the percentage of polarization.   B is a double-image prism of quartz, C the camera with quartz lens, and D the exhausted tube containing the mercury vapour, a cross-section of the illuminated end of which is shown at E.

Fig. 1.
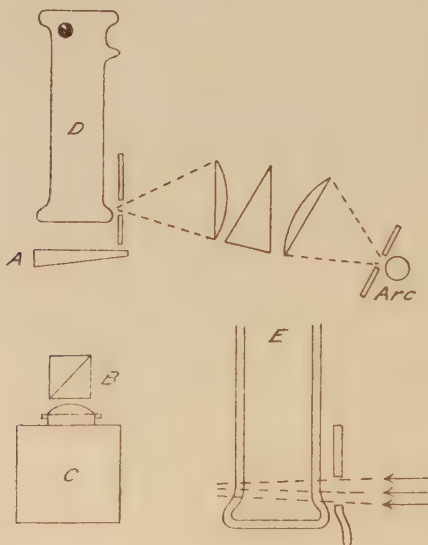
Photographs of the polarization fringes reproduced as negatives on Plate IX. show clearly that they are distinct quite up to the wall of the tube, where the exciting radiation enters, and that they are of equal visibility all along the column of excited vapour. Photographs were also made through the side of the tube, with the exciting radiation passing down its axis, and similar results obtained. The fringes are more distinct when the opposite end of the quartz tube and the contained drop of mercury are cooled with a bath of ice and salt. At room temperature there is much secondary resonance radiation between the primary beam and the observation window, and this undoubtedly reduces the visibility of the fringes. I believe, however,

that the actual pressure of the mercury vapour also influences the percentage of polarization, for it was found that it fell rapidly if any other gas was admitted to the tube, helium at 6 mms. pressure destroying practically all traces of polarization. In cases where the vapour was excited by polarized light (electric vector vertical as seen from the camera) the visibility of the fringes indicated nearly complete polarization. With unpolarized excitation the visibility was somewhat less, though the fringes were still very distinct.

These results show that in no case is the polarization complete, for if it were, the visibility of the fringes would be the same with unpolarized as with polarized excitation.

The fact that the fringes are more distinct when the density of the mercury vapour is lowered by refrigeration of one end of the tube, indicates that molecular impacts probably have something to do with the depolarization. Admixture of other gases at low pressures reduces the fringe visibility, and finally destroys all trace of polarization, though the intensity of the resonance radiation may be quite unimpaired.

Air at 1 cm. pressure practically destroys the polarization while it is still pronounced in hydrogen at the same pressure. Curiously enough, however, the intensity of the resonance radiation of mercury in hydrogen is less than half of its value in air. This is a marked exception to the law which was found by Wood and Franck, in the case of iodine fluorescence, namely that gases diminished the intensity in proportion to their electro-negative character, air being much more destructive of the fluorescence than hydrogen.

The results obtained with air, hydrogen, argon, and helium are given in the following table.

	Intensity of Resonance Rad.	Polarization fringes.
Air 0·65 mm.	2·5	strong.
Hydrogen 0·65 mm.	1·	strong.
Air 4 mm.	1·	nearly gone.
Hydrogen 4 mm.	0·5	faint.
Air 1 cm.	0·5	gone entirely.
Hydrogen 1 cm.	0·2	a trace.
Air 5 mm.	3·	a trace.
Argon 5 mm...............	10·	gone.
Argon 3 mm.	5·	faint.
Helium 2 mm.	4·	faint.
Helium 6 mm.	10·	gone.

The decrease in the percentage of polarization caused by the admixture of small quantities of other gases, and its low value with pressures of mercury vapour much in excess of the value which it has at room temperature, is of considerable theoretical importance, for if there is a brief interval of time between the absorption of energy by the mercury molecule and its re-emission, and the depolarizing factor is a rotation of the molecule, it is probable that the magnitude of these quantities can be experimentally determined by making a careful and exact determination of the percentages of polarization corresponding to different pressures and temperatures. This matter is now under investigation.

In the case of the green fluorescent light emitted by mercury vapour at higher densities, when stimulated by ultraviolet light in the spectral range well below the 2536 line, the time interval between absorption and emission is so long that it can be shown experimentally by methods which I described in the Proc. Roy. Soc. for 1921. The vapour was distilled in an exhausted quartz tube in the form of an inverted U, the rising column of vapour remaining dark within the narrow beam of exciting rays, but bursting into fluorescence at a distance of several millimetres above this region, in the form of a pointed green flame, concave on its under side. The resonance radiation, excited by a narrow pencil of rays of wave-length 2536, traversing the U-tube, was confined wholly within the illuminated region, showing that the time interval between absorption and emission, if it exists, is very much briefer in this case.

The action of argon and helium in augmenting the intensity of the resonance radiation is of considerable interest. With argon at 0·5 mm. pressure the intensity of the resonance radiation is noticeably greater than in a highly exhausted tube. The same is true with argon at 60 mms. pressure, the intensity being more than doubled by the presence of the gas.

Helium at 330 mms. increased the intensity to fully four times its value in an exhausted tube. At first sight this seems very surprising in view of the fact that in all cases previously observed, so far as I know, the introduction of a foreign gas decreases the intensity of the fluorescence, for example the introduction of helium or argon into fluorescing iodine vapour. A little consideration, however, made it seem probable that, in this case, the argon and helium

increased the spectral range of the frequencies to which the mercury molecules could respond, in other words broadened the 2536 absorption line of the mercury vapour, which was thus able to divert as resonance radiation more energy from the exciting beam. In vacuo, the mercury vapour abstracts only the " core " of the very narrow 2536 line : in helium or argon practically the entire energy of the line may be diverted. This was tested in the following way.

The 2536 light from the monochromator was passed through a highly exhausted quartz tube containing a drop of mercury. This removed the " core " of the line, the light emerging from the tube causing no resonance radiation in a second exhausted mercury tube, though still nearly as intense as the original beam. Helium was now admitted to the second tube, and intense resonance radiation immediately observed, proving that the mercury vapour in helium will resonate to

Fig. 2.
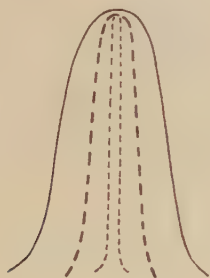
the frequencies to the right and left of the core of the exciting line, as indicated in fig. 2. Here we have the intensity curve of the 2536 exciting line represented by an unbroken line, the portion abstracted by mercury vapour in vacuo by a dotted line, and the portion taken up by mercury in argon or helium by the line of dashes.

CIII. *Electric Fields due to the Motion of Constant Electromagnetic Systems.* By S. J. BARNETT [*].

§1. IN this article Maxwell's equation for the electromotive intensity, together with a theorem derived from it in §600 of his Treatise, and hitherto but little used, will be applied to the investigation of a number of simple but fundamental fields. Some of these fields are well known, some are new; all of them are worth considering from the standpoint of Maxwell's theorem.

Consider an electromagnetic system B which has a velocity v relative to fixed axes C. Let E, E', B, B', ψ, ψ', and A, A' denote the electromotive intensities, magnetic inductions, electric potentials, and vector potentials observed at a point P fixed in C by one at rest in C and by one moving with the system B, respectively. The electromotive intensity at P, or the force per unit charge upon an infinitesimal charged body fixed at P, is, in Gaussian units,

$$E = -\frac{1}{c}\frac{\partial A}{\partial t} - \nabla\psi \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

to the fixed observer. To the moving observer it is

$$E' = -\frac{1}{c}\left(\frac{\partial A'}{\partial t}\right) - \nabla\psi' - \frac{1}{c}[vB'], \quad . \quad . \quad . \quad (2)$$

where

$$\left(\frac{\partial A'}{\partial t}\right) = \frac{\partial A}{\partial t} + (v\nabla)A. \quad . \quad . \quad . \quad . \quad (3)$$

The electromotive intensity at P, if fixed to the moving system B, is

$$F = E + \frac{1}{c}[vB] = -\frac{1}{c}\frac{\partial A}{\partial t} - \nabla\psi + \frac{1}{c}[vB] \quad . \quad (4)$$

to an observer in C, and

$$F' = E' + \frac{1}{c}[vB'] = -\frac{1}{c}\left(\frac{\partial A'}{\partial t}\right) - \nabla\psi' \quad . \quad . \quad . \quad (5)$$

to an observer moving with B.

Assuming a principle of relativity according to which

$E = E'$, $B = B'$ (so that also $A = A'$), Maxwell has shown that the motion produces (to an observer fixed in C) an electric field whose polar part is derivable from the potential

$$\phi = \psi - \psi' = \frac{1}{c}(Av). \quad . \quad . \quad . \quad . \quad (6)$$

From the assumptions made it is clear that the result is only an approximation; but, as pointed out by Larmor *, who has given another derivation of (6), the error is only of the second order in v/c.

For a constant electromagnetic system,

$$\left(\frac{\partial A'}{\partial t}\right) = \left(\frac{\partial A}{\partial t}\right) = 0, \text{ and therefore } \frac{\partial A}{\partial t} = -(v\nabla)A. \quad (7)$$

If also the system is unelectrified (in B), as will be assumed henceforth,

$$\nabla \psi' = 0, \text{ and } \nabla \psi = \frac{1}{c}\nabla(Av). \quad . \quad . \quad . \quad . \quad (8)$$

In this case

$$E = \frac{1}{c}(v\nabla)A - \frac{1}{c}\nabla(Av) = E' = \frac{1}{c}[Bv]. \quad . \quad (9)$$

The electromotive intensity of the field produced by the motion is thus given completely by $\frac{1}{c} \times$ the vector product of B and v.

§2. The vector potential at a point distant r from an element of space (volume, surface, or length) $d\tau$ in which the current density (volume, surface, or line) is i is defined as

$$A = \int \frac{i \, d\tau}{cr}. \quad . \quad . \quad . \quad . \quad (10)$$

We have also the relation

$$\text{curl } A = B, \text{ or } \int (A \, dl) = \int (B \, dS), \quad . \quad . \quad (11)$$

where dl is the element of length of a closed curve bounding the surface whose area is S. From one or both of these equations we can always determine A, and hence $\phi = \frac{1}{c}(Av)$.

If σ denotes the electric density (volume, surface, or line)

* J. Larmor, Phil. Mag. xvii. p. 1, Jan. 1884.

at any point of the system where the current density is i, we obtain from (6) and (10) the relation

$$\phi = \int \frac{\sigma d\tau}{r} = \frac{1}{c^2} \int \frac{(iv)}{r} d\tau,$$

or $$\sigma = \frac{(iv)}{c^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

This result was obtained from Clausius's theory in 1880 by E. Budde [*]; the corresponding result with the correction for the second-order term in v/c was obtained by Lorentz [†] in 1895, and by Silberstein [‡], from Minkowski's equations, in 1914.

From (12) it is easily shown that in the general case the electric moment Q of the charges developed by the motion is equal to $1/c$ multiplied by the vector product of the velocity and the magnetic moment M; that is,

$$Q = \frac{1}{c} [vM], \quad . \quad . \quad . \quad . \quad . \quad (13)$$

of which a number of special cases will be found below [§].

§ 3. *Two infinite plane parallel current sheets* B *and* C *with equal and opposite currents,* I *per unit length, in motion parallel to the stream-lines* (fig. 1). From (10) it is clear

Fig. 1.
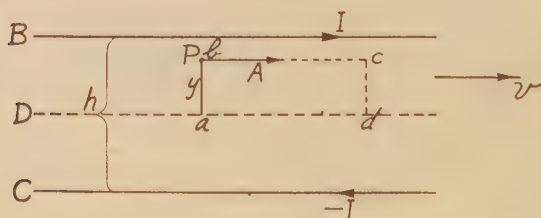
that $A = 0$ over the central plane D parallel to the sheets, and that elsewhere it has the direction of the current in the nearer sheet. To obtain the vector potential at a point P

* E. Budde, *Ann. der Phys.* x. p. 553 (1880).

† H. A. Lorentz, *'Versuch,'* 1895, § 25.

‡ L. Silberstein, *'The Theory of Relativity,'* 1914, p. 272.

§ Equations (6) and (12) have both very recently been derived for a special case by W. F. G. Swann, who does not refer to the earlier work by Maxwell and the others mentioned above. He has also obtained (13) for the special case of a doublet, but with the wrong sign. See Phys. Rev. xv. p. 365 (1920).

between the sheets and distant y from D in the direction of the sheet B, in which v and I have the same direction, we may apply (11) to the area $abcd$. We thus get $A \times bc = B \times bc \times y = \frac{4\pi I}{c} \times bc \times y$; so that

$$A = By = \frac{4\pi I}{c} y.$$

Similarly, we find for the region above B,

$$A = \frac{4\pi I}{c} \times \frac{h}{2},$$

and for the region below C,

$$A = -\frac{4\pi I}{c} \times \frac{h}{2}.$$

Between the plates

$$\phi = \frac{1}{c}(Av) = \frac{4\pi I}{c^2} vy;$$

so that

$$-\frac{1}{c} \nabla(Av) = -\frac{4\pi I}{c^2} v = \frac{1}{c} [Bv]$$

is uniform and directed from B toward D. The surfaces of C and D have thus the charges $+\frac{Iv}{c^2}$ and $-\frac{Iv}{c^2}$ per unit area, in agreement with the general relation (12). Outside the region between the sheets $\phi = \pm \frac{4\pi I vh}{2c^2}$ and $-\nabla\phi = 0.$ Here $\frac{\partial A}{\partial t} = -(r\nabla)A = 0$, so that the field is purely polar and $-\frac{1}{c}\nabla(Av)$ is the total electric intensity.

It is easy to see that if M denotes the magnetic moment of the system, and Q its electric moment,

$$Q = \frac{1}{c}[vM].$$

The charges developed by the motion, being themselves in motion with velocity v, produce a magnetic field in the direction of the original field and with intensity

$$h = \frac{4\pi}{c} \cdot \frac{Iv}{c^2} \cdot v = H \frac{v^2}{c^2},$$

thus showing, in conformity with Larmor's statement, that the adoption of Maxwell's method introduces an error of the

second order in v. We may proceed similarly in the other cases which follow.

§ 4. *Two infinite parallel wires* B *and* C, *with currents* I *and* $-$I, *in motion parallel to their lengths.* For a point distant r_1 from B (whose current I is in the direction of r) and r_2 from C, we have

$$A = \frac{1}{c} \int \left(\frac{1}{r_1} - \frac{1}{r_2} \right) d\tau$$

and

$$\phi = \frac{1}{c}(Av) = \frac{vI}{c^2} \int \left(\frac{1}{r_1} - \frac{1}{r_2} \right) d\tau,$$

so that the electric field produced is that terminated by equal and opposite charges on the wires, the charges per unit length upon B and C being $\frac{vI}{c^2}$ and $-\frac{vI}{c^2}$. Here also $\left(\frac{dA}{dt} \right) = \frac{\partial A}{\partial t} = 0$, and $E = \frac{1}{c}[Br]$. Furthermore, as in the last section,

$$Q = \frac{1}{c}[vM].$$

§ 5. Similar results are obtained for *two circular cylindrical coaxial conductors traversed by equal and opposite currents and in motion parallel to the axis.*

§ 6. *An infinite uniform cylindrical current sheet, with current* I *per unit length of the axis, in motion normal to the*

Fig. 2.
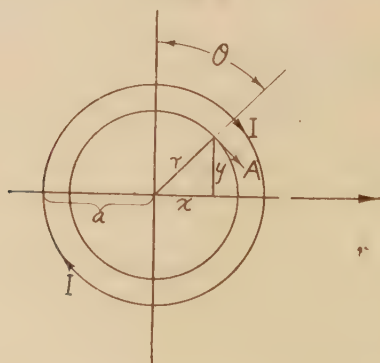
axis (fig. 2). Both inside and outside the cylinder the vector potential is evidently constant in magnitude over each coaxial cylinder and is tangential to the cylinders in planes normal

to the axis. For a cylinder of radius r within the current sheet, we have

$$2\pi r \times A = \pi r^2 B = \pi r^2 \times \frac{4\pi I}{c},$$

so that

$$A = \frac{rB}{2} = \frac{2\pi I}{c} \cdot r$$

and

$$\frac{1}{c}(Av) = \frac{2\pi Iv}{c^2} \cdot r \cos\theta = \frac{2\pi Iv}{c^2} y.$$

Thus

$$-\frac{1}{c}\nabla(Av) = -\frac{2\pi Iv}{c^2} = \frac{1}{2c}[Bv]$$

is uniform within the sheet and is directed downward in the figure.

The surface density of the charge which terminates the uniform field with intensity $-\dfrac{1}{c}\nabla(Av)$ is

$$\sigma_1 = \frac{2\pi Iv}{4\pi c^2}\cos\theta = \frac{Iv}{2c^2}\cos\theta.$$

The charges per unit length terminating this field on the two hemi-cylinders are

$$\pm q_1 = \sigma_1{}_{(\theta=0 \text{ or } \pi)} \times 2a = \frac{vIa}{c^2}.$$

The electric moment Q_1 of these charges, positive on the upper half and negative on the lower half, is evidently

$$Q = \int_{-a}^{a} 2y \cdot \frac{Iv}{2c^2} \cdot dx = \frac{\pi a^2 Iv}{2c^2},$$

and is directed upward in the figure. The magnetic moment of the system, per unit length, is

$$M = \frac{\pi a^2 I}{c}$$

directed into the paper in the figure. Thus

$$Q_1 = \frac{1}{2c}[vM].$$

Outside the sheet $(r > a)$,

$$A = \frac{2\pi Ia^2}{cr}$$

and

$$\phi = \frac{1}{c}(Av) = \frac{2\pi Ia^2 v\cos\theta}{c^2 r}.$$

This is the potential due to a line doublet situated on the axis of the sheet with electric moment

$$Q = \frac{\pi I a^2 v}{c^2}$$

per unit length and directed upward in the figure.

Thus $$Q = \frac{1}{c}[vM].$$

The outward radial component of the intensity at the surface of the cylinder due to the doublet of moment Q is $\frac{2Q}{c^2 a^2}\cos\theta = \frac{2\pi I v}{c^2}\cos\theta$. Thus the surface density of the part of the charges on the cylinder terminating the outer field is

$$\sigma_2 = \frac{Iv}{2c^2}\cos\theta = \sigma_1$$

and its moment $Q_2 = Q_1$. Thus the total electric moment per unit length of the cylinder is

$$Q = 2Q_1 = \frac{1}{c}[vM].$$

The total charge per unit length on the upper and lower hemi-cylinders is

$$q = 2q_1.$$

In addition to the electric intensity $-\frac{1}{c}\nabla(Av)$, there is also the solenoidal intensity

$$-\frac{1}{c}\frac{\partial A}{\partial t} = \frac{1}{c}(v\nabla)A,$$

which does not vanish. Inside the cylinder

$$\frac{1}{c}(v\nabla)A = \frac{1}{c}(v\nabla)A_y = \frac{v}{c}\frac{\partial A_y}{\partial x} = -\frac{2\pi I v}{c^2}$$
$$= \frac{1}{c}\nabla(Av) = \frac{1}{2c}[Bv],$$

so that the total intensity is $\frac{1}{c}[Bv]$.

Outside the cylinder

$$\frac{1}{c}\{(v\nabla)A\}_x = \frac{2\pi I a^2}{c^2}\frac{\partial}{\partial x}\left(\frac{y}{r^2}\right) = -\frac{4\pi I a^2 v}{c^2}\frac{xy}{r^4}$$

and

$$\frac{1}{c}\{(v\nabla)A\}_y = -\frac{2\pi I a^2}{c^2}\frac{\partial}{\partial x}\left(\frac{x}{r^2}\right) = -\frac{2\pi I a^2 v}{c^2}\frac{(y^2 - x^2)}{r^2}.$$

These are the component intensities due to an axial line doublet with moment equal and opposite to that already obtained. Thus

$$\frac{1}{c}(r\nabla)A - \frac{1}{c}\nabla(Ar) = 0 = \frac{1}{c}[Bv].$$

The field whose intensity is $\frac{1}{c}(r\nabla)A$ is thus a circuital field without divergence whose flux cancels that due to the charges in the region outside the cylinder and doubles that due to the charges inside.

§ 7. *A spherical current sheet, with current* I *per unit length of the diameter along the magnetic axis, in motion normal to the magnetic axis.* The magnetic field within the sphere is known to be uniform, normal to the planes of the stream lines, with induction

$$B = \frac{8\pi I}{3c}.$$

The vector potential is evidently tangential to circles centred on the magnetic axis and normal to the magnetic axis.

Within the sphere, for a cylinder of radius r,

$$A = \frac{4\pi I}{3c} r$$

and

$$\phi = \frac{1}{c}(Av) = \frac{4\pi Iv}{3c^2} r \cos\theta = \frac{4\pi Iv}{3c^2} \cdot y.$$

Thus

$$-\frac{1}{c}\nabla(Av) = -\frac{4\pi Iv}{3c^2} = \frac{1}{2c}[Bv].$$

That part of the surface density due to the charges which terminate this uniform field is evidently

$$\sigma_1 = \frac{Iv}{3c^2}\cos\theta.$$

The total charge (terminating the uniform field) on each hemisphere is

$$\pm q_1 = \pi a^2 \cdot \sigma_{1(\theta=0 \text{ or } \pi)} = \pm\frac{\pi a^2 Iv}{3c^2},$$

and the moment of these charges is

$$Q_1 = \int_0^\pi \sigma_1 \cdot 2\pi a \sin\theta \cdot a\,d\theta \cdot a\cos\theta = \frac{1}{3}\cdot\frac{4}{3}\frac{\pi a^3 Iv}{c^2}.$$

The magnetic moment of the sphere is

$$M = \frac{4\pi a^3 I}{3c^2},$$

so that, with proper attention to signs,

$$Q_1 = \frac{1}{3c}[vM].$$

Outside the sphere the magnetic field is that of a point doublet of moment M at the centre of the sphere. The vector potential is thus

$$A = -\left[M\nabla\frac{1}{r}\right];$$

so that

$$\phi = \frac{1}{r}(Av) = -\frac{1}{c}\left(\left[M\nabla\frac{1}{r}\right]v\right) = -\frac{1}{c}\left(\nabla\frac{1}{r}\cdot[vM]\right),$$

which is the potential of an electric point doublet with moment

$$Q = \frac{1}{c}[vM]$$

at the centre of the sphere.

The outward radial component of the intensity at the surface of the sphere due to the doublet of moment Q is $\frac{2Q}{a^3}\cos\theta$. Thus the surface density of the part of the charges terminating the outer field is

$$\sigma_2 = \frac{Q}{2\pi a^3}\cos\theta = \frac{2}{3}\frac{Iv}{c^2}\cos\theta = 2\sigma_1.$$

The total charges upon the upper and lower hemispheres terminating the outer field are thus

$$\pm q_2 = 2q_1 = \frac{2}{3}\frac{\pi a^2 Iv}{c^2},$$

and the moment of these charges is

$$Q_2 = 2Q_1.$$

The total surface density upon the two hemispheres is thus

$$\sigma = \sigma_1 + \sigma_2 = \frac{Iv}{c^2}\cos\theta,$$

in conformity with (12). The total charges upon the hemispheres are

$$\pm q = \pm (q_1 + q_2) = \pm \frac{\pi a^2 I v}{c^2}.$$

and the moment of the charges is

$$Q_1 + Q_2 = Q = \frac{1}{c}[vM].$$

Within the sphere.

$$\frac{1}{c}(r\nabla)A = \frac{1}{c}\{(r\nabla)A\}_y = \frac{v}{c}\frac{\partial A_y}{\partial x} = -\frac{4}{3}\frac{\pi I v}{c^2} = \frac{1}{2c}[Bv],$$

which is equal to the intensity $-\frac{1}{c}\triangle(Av)$.

Outside the sphere $-\frac{1}{c}\nabla(Av)$ is not everywhere even in the same direction as the total intensity $\frac{1}{c}[Bv]$. $\frac{1}{c}(r\nabla)A$ is again the intensity of a solenoidal field without divergence whose flux through the sphere doubles the intensity inside and makes it everywhere equal to $\frac{1}{c}[Bv]$.

§ 8. *Two cylindrical coaxial magnetic poles with a radial magnetic field between them, and in motion parallel to the*

Fig. 3.
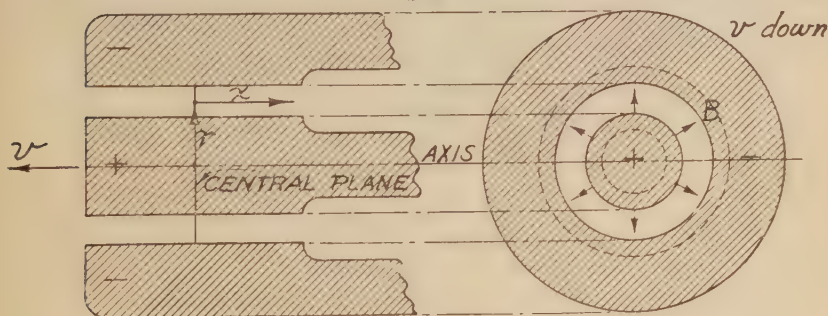
axis (fig. 3). In this case the vector potential vanishes over the central symmetrical plane of the field. The lines of vector potential are circles, centred on the axis and normal thereto. The magnetic flux across a cylinder of radius r and length x, one end in the symmetrical plane, is $2\pi r x B =$

$2\pi r A$; so that $A = Bx = \frac{Cx}{r}$ where C is a constant. To one

looking in the direction of v the circles are right-handed behind the central plane, left-handed in front. In this field $\phi = \frac{1}{c}(Av) = 0$, the electric field is circuital, without charges,

and
$$E = \frac{1}{c}\frac{\partial A}{\partial t} = \frac{1}{c}(v\nabla)A = \frac{1}{c}[Bv].$$

The lines of electric intensity are left-handed circles centered on the axis in planes normal thereto.

§ 10. The introduction of conductors (with negligible magnetic susceptibility) moving rigidly with the system B, and uncharged in B, does not affect the electric field produced by the motion in any way at points not within the material of the conductors. For it does not alter $A' = A$, and therefore does not alter $-\nabla(Av)$ or $(v\nabla)A$. Thus, if in the field of § 3 a closed conducting box, moving with the current sheets, were to surround the observer in C, the electric intensity observed would remain unaltered. It is therefore unnecessary to inquire whether the substances in B are conducting or insulating. If they are magnetic, the vector potentials of the magnetons will have to be taken account of.

Another way of looking at the matter is, of course, this : The introduction of the conductors and the consequent induction of electric charges upon them may greatly affect the intensity at P due to the motion. But the effect of these charges is entirely neutralized by the distribution produced by the motional intensity $\frac{1}{c}[vB]$ acting in and upon the conductors themselves. Within the material of the conductors forming a part of the system B, or rigidly moving with B, the total electromotive intensity is

$$F = \frac{1}{c}(v\nabla)A - \frac{1}{c}\nabla(Av) + \frac{1}{c}[vB] = \frac{1}{c}[Bv] + \frac{1}{c}[vB] = 0.$$

§ 11. In the case of *a conducting cylinder or sphere with uniform intensity of magnetization* I *in motion normal to the magnetic axis,* all the formulæ, for points outside the system, are exactly similar to those for the cylindrical and spherical current sheet, but I has now the meaning attached to it here instead of that of §§ 7 and 8. Within the material of the system the electromotive intensity is zero.

If we apply to the magnetic elements, or magnetons, the same method previously applied to the whole current sheet, we evidently find that each magneton with component magnetic moment μ parallel to the magnetic axis becomes so

charged by the motion as to have the electric moment $\frac{1}{c}[v\mu]$. Thus the sphere becomes electrically polarized, with polarization, or electric moment per unit volume, given by

$$P = \frac{1}{c}\Sigma[v\mu] = \frac{1}{c}[vI],$$

where the summation extends over the unit volume. The field of this polarization gives the polar part of the external field, and, together with the internal part of the solenoidal field, just balances the motional intensity $\frac{1}{c}[vB]$ inside the system : or we may consider that the effect of the polarization is neutralized by that of an equal and opposite polarization due to the charges induced in the parts of the conductor adjacent to the individual magnetons, and that the distribution produced by the motional intensity gives the polar part of the external field, which, together with the solenoidal part, just balances the motional intensity *.

§ 12. In the case of *two similar infinitely long magnets with rectangular cross-sections placed parallel with opposite poles facing one another symmetrically and in motion parallel to their lengths and normal to the lines of induction of the magnetic field*, the vector potential is evidently zero over the central plane parallel to the motion and normal to the pole faces. It is everywhere parallel to the velocity : and its direction is related to that of the central part of the magnetic field exactly as in the case of § 3. Its magnitude is independent of the coordinate parallel to the length. Thus

$$\frac{\partial A}{\partial t} = -(v\nabla)A = 0 ;$$

and the total intensity outside the substance of the magnets, viz.,

$$E = -\frac{1}{c}\nabla(Av) = \frac{1}{c}[Bv],$$

lies in planes normal to the motion, or is two-dimensional like the magnetic field. Within the magnets themselves the total electromotive intensity is zero, and, as in § 11, there is an electric polarization $P = \frac{1}{c}[vI]$ at points where the intensity of magnetization is I.

* See E. Budde, *l. c. ante,* and W. F. G. Swann, *l. c. ante.*

Experiments made by the author * in 1918 with a screened condenser placed symmetrically between two much larger parallel magnets in motion like those above, are consistent with the view that the field is polar, as required by the theory. It is immaterial whether the intensity is calculated from $[Bv]$, as was done by the author, or whether it is calculated from $-\nabla(Ar)$; or we may consider that the effect of the polarization is exactly neutralized by that of the equal and opposite polarization due to the charges induced on the parts of the conductor adjacent to the individual magnetons, and that the net electric field remaining is due to the electric displacement produced by the motional intensity $\frac{1}{c}[vB]$ †.

§ 13. Maxwell's theorem $\phi = \frac{1}{c}(Ar)$ cannot in general be applied immediately to the case of an electromagnetic system forming a solid or surface of revolution about an axis of symmetry and in steady rotation about this axis ‡, although it may be applied to each element of which the system is composed and which has its own linear velocity and vector potential.

The electric field surrounding such a body, rotating either in a neutral region or in an impressed magnetic field directed along the axis of rotation or symmetrical about this axis, can be determined at once from equations (1) and (4). In this case $\frac{\partial A}{\partial t} = 0$, so that the field is purely polar, derived from the potential ψ, which can be calculated from the motional intensity $\frac{1}{c}[vB]$, and is due to the charges produced in and on the rotating electromagnetic system.

Jochmann [*] and Larmor [†] have both worked out the general form of the solution for the case in which the system is a conducting solid of revolution and the magnetic induction symmetrical about the axis of rotation, and have given the details for the case in which the solid is a sphere and the induction uniform.

The potential at a point distant $r = (> a)$ from the centre of the sphere and with co-latitude θ is

$$\psi = C\,\frac{a}{cr} - \frac{1}{3}\,\omega B\,\frac{a^5}{cr^3}\left(1 - \frac{3}{2}\sin^2\theta\right) + \frac{1}{3}\frac{\omega B a^3}{cr},$$

where ω is the angular velocity and C is a constant. When the sphere remains insulated from the initiation of the motion, $C = -\dfrac{\omega B a^2}{3}$, and only the middle term of ψ remains. When the pole is earthed, $C = 0$ [‡]. The sphere is uniformly charged within, the volume density being $-\dfrac{\omega B}{2\pi c^2}$. Still further details are given by Jochmann.

§ 14. If the rotating body is magnetic, it is electrically polarized by the motion, the polarization at any point where the intensity of magnetization is I being $P = \dfrac{1}{c}\,[(\omega r)I]$. On account of the conductivity of the body, however, an equal and opposite polarization is produced by electric induction, and no gross effect of the polarization appears. The existence of this polarization, but with the wrong sign, together with its neutralization by conduction, has been pointed out by Swann (*l. c. ante*).

§ 15. Experiments by the author [§], by E. H. Kennard [||], and by G. B. Pegram [¶], in which a screened cylindrical condenser was placed coaxially in the field of the rotating system of § 13, short-circuited, and tested for charge when at rest and when rotating with the electromagnetic system,

[*] E. Jochmann, Crelle's Journal, xxxvi. p. 329 (1863); also Phil. Mag. xxviii. p. 347 (1864).

[†] J. Larmor, Phil. Mag. xvii. p. 1, Jan. 1884. See also, J. Larmor, Roy. Soc. Phil. Trans. A. 186, p. 695 (1895).

[‡] Recently the above results for the sphere have been published by Swann, *l. c. ante,* who does not refer to previous investigators.

[§] S. J. Barnett, Phys. Rev. xxxv. p. 323 (1912).

[||] E. H. Kennard, Phil. Mag. xxxiii. p. 179 (1917); see also, for a related experiment, Phil. Mag. xxiii. p. 937 (1912).

[¶] G. B. Pegram, Phys. Rev. x. p. 595 (1917).

are all in accord with Jochmann and Larmor's theory of § 13. That is, when the condenser remains at rest, the charge is zero, because the field is polar, with external charges; when in motion, the charge is due to the intensity $\frac{1}{c}[(\omega r)B]$ in the conductors. The application of Maxwell's equations (1) and (4), as used by Larmor, to these experiments was discussed in detail by Pegram in 1917. In many experiments from Faraday to the present time the potential difference between different circles of latitude on the rotating body has been measured with a galvanometer or an electrometer; recently the charge on the sphere of § 13, with axis earthed, and surrounded by a concentric sphere, has been measured by Swann *. All the results are in agreement with the theory of Jochmann, Maxwell, and Larmor.

§ 16. In the case of a symmetrical electrical circuit rotating about its axis of symmetry, the rotation produces no rotating effect on the tubes of induction of the magnetic field, as indicated in § 13 and shown very simply by Pegram. Applying this result to the Ampèreian vortices of a magnet, it was Pegram's idea that when the magnet rotated, each vortex carried its lines of induction in translation, but that the lines of induction of the vortex did not share the rotating motion of the magnet †. This idea has recently been discussed in detail by Swann (*l. c. ante*), but it is not new. It is only in this sense that the "moving line" theory of unipolar induction is true.

We have already seen that so far as the net external result is concerned, the effect of the polarization due to the translatory motion of the magnetons may be considered neutralized by that of induced charges. The intensity e at any point P of any plane through the axis of rotation due to the motion of an element of the magnet with magnet moment m and velocity $v = [\omega r]$ is

$$e = -\nabla(ar) + (v\nabla)a,$$

* W. F. G. Swann, Phys. Rev. xix. p. 38 (1922).
† This is not discussed in detail by Pegram, but was clearly in his mind and is an immediate corollary of the case of the simple coil. After Pegram's paper was read at Cleveland, but in less complete form than that of the printed article, I stated to him that in the case of a magnet I thought the fundamental way to consider the matter was to assume that each electron orbit carried its lines of induction with it. He immediately assented, and remarked in addition that the rotatory part of the motion was not to be considered.

while the intensity e' at the same point due to the corresponding element with (equal) moment m' symmetrically situated on the opposite side of the plane is evidently

$$e' = -\nabla av - (v\nabla)a.$$

Thus at any point P the solenoidal parts of the fields due to corresponding elements cut one another out in pairs, while the polar parts of the intensities are equal and additive. The polarization, due to the polar parts, being neutralized by induction, the total gross effect of the motion of the vortices vanishes.

§ 17. From the fundamental equations of electromagnetic theory as developed by Cohn [*], and later by Minkowski[†], and still later by Einstein and Laub [‡], a general expression has been obtained for the polarization produced in an insulator by its motion in a magnetic field. If K denotes the dielectric constant of the medium, μ its permeability, I its intensity of magnetization, B the magnetic induction, and r the velocity, the formula for the polarization, in the approximate form obtained by M. Abraham [§], is

$$P = \frac{K}{4\pi c}\left(1 - \frac{1}{\mu K}\right)[vB].$$

This polarization is easily shown to consist of two distinct parts: one, P_1, produced by the motional intensity $\frac{1}{c}[vB]$ acting on the moving part of the insulator; the other, P_2, due to the motion of the magnetons.

On the theory of Lorentz and Larmor the æther is at rest, so that only the electrical fraction $(K-1)/K$ of the insulator is in motion. Hence

$$P_1 = \frac{K-1}{4\pi}\frac{1}{c}[vB].$$

This result has been fully confirmed by experiments made on air in 1901 by Blondlot [‖], on ebonite in 1904 by H. A.

[*] E. Cohn, *Ann. der Phys.* vii. p. 29 (1902).
[†] H. Minkowski, *Gott. Nachr. Math. Phys. Kl.* 1908, p. 53.
[‡] A. Einstein and J. Laub, *Ann. der Phys.* xxvi. p. 532 (1908).
[§] M. Abraham, 'Theorie der Elektrizitaet,' ii. § 38 (1908).
[‖] R. Blondlot, *C. R.* cxxxiii. p. 778 (1901).

Wilson *, and on rosin, sulphur, and ebonite in the interval 1902–8 by the author †.

From Maxwell's theorem

$$P_2 = \frac{1}{c}[vI] = \frac{1}{4\pi c}\left(1 - \frac{1}{\mu}\right)[cB].$$

Adding together the expressions for P_1 and P_2 we obtain very simply the expression for P given above as hitherto derived only on the basis of the Cohn-Minkowski equations ‡.

The Carnegie Institution
  of Washington,
Department of Terrestrial Magnetism,
      April, 1922.

CIV. *On the Temperature Ionization of Elements of the Higher Groups in the Periodic Classification.* By MEGH NAD SAHA, *D.Sc., F.I.P., Gurnprosad Singha Professor of Physics, University of Calcutta, India* §.

I.

THE theory of the temperature ionization of gases and its application to problems of radiation and astrophysics was given by the present writer in a number of papers published during last year. In these papers the theory was limited to the ionization of gas consisting of atoms of a

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* H. A. Wilson, Roy. Soc. Phil. Trans. A. 1904, p. 121.   Wilson considered his experiments to prove that the motional electric *intensity* or *electromotive force* was proportional to $(K-1)$, which is not correct. According to all theories, the motional intensity is independent of the medium and equal to $\frac{1}{c}[cB]$; while on the theory of Larmor and Lorentz, the resulting *polarization* is proportional to $(K-1)$, the result supported by the experiments.

† S. J. Barnett, Phys. Rev. xxvii. p. 425 (1908).

‡ The permeability μ differs from unity so slightly for all insulators that it is impossible at present to distinguish experimentally between P and P_1.   By embedding a large number of small steel spheres in wax, however, M. Wilson and H. A. Wilson (Proc. Roy. Soc. A. lxxxix. p. 99 (1914)) formed a composite dielectric whose mean permeability, for large volumes, was much greater than unity.   On the assumption that this procedure is justifiable, the results of experiments which they made on the electric effect of moving the composite substance in a magnetic field support the above equation for P.   M. and H. A. Wilson concluded that their results therefore supported the (Einstein-Minkowski) principle of relativity.   As shown in §17, however, the result follows from Maxwell's theorem based on a much older, though less exact, relativity principle.

§ Communicated by the Author.

single kind. Recently E. A. Milne and Henry Norris Russell * have extended the theory to mixtures of elements. By a comparison of the sun-spot and the solar spectrum, Russell finds that the predictions of the theory with reference to the relative intensity in the hotter and the cooler spectrum of lines associated with ionized and non-ionized atoms are found to be in general agreement with the facts. Russell has also shown that the temperature of the sun and the sun-spot comes out to be much more in accord with the figures obtained from general intensity measurements when mixtures of different elements are considered instead of one single element.

But discrepancies have also been pointed out by Russell, which suggests the need of some modification. The nature of these discrepancies may be grasped from the following.

Let I_1 and I_2 be the ionization-potentials of elements A and B. Then at a definite temperature and pressure the ratio of the degrees of ionization of A and B is given by the equation

$$\log \frac{K_1}{K_2} = \log\left(\frac{x_1}{1-x_1}\right) - \log\left(\frac{x_2}{1-x_2}\right) = 5036 \frac{I_1 - I_2}{T}$$

where x_1, x_2 are the fractions ionized, I_1, I_2 are expressed in volts.

If $I_1 = I_2$, x_1 should equal x_2.

That this is not the case is shown from the fact that sodium and barium have got practically the same ionization potential ($5\cdot11$ and $5\cdot12$ volts respectively), yet both in the sun and in the sun-spot, barium is a good deal more ionized than sodium. The resonance line of Ba, $\lambda = 5535\cdot93$, is absent or very faint both in the solar and the spot spectrum, and it is represented only by the enhanced lines (Ba$^+$, $\lambda = 4934\cdot07$, $4554\cdot04$), which shows that barium is completely ionized not only in the sun but also in the spot. The resonance lines of sodium, $\lambda = 5889\cdot97$, $5895\cdot94$, on the other hand, are very prominent in the solar spectrum, and are greatly intensified in the spot, which shows that in the sun a large percentage of sodium is unionized, and in the spot the percentage increases owing to a lowering of temperature.

What has been said of sodium and barium admits of a widegoing generalization, viz. the alkaline earths are, as a rule, much more strongly ionized than their ionization potential would indicate. The behaviour of the alkalies is

* Milne, The Observatory, Sept. 1921 ; Russell, The Astrophysical Journal, March 1922.

normal, and we shall see later on that, as a rule, *elements belonging to the higher groups are more easily ionized than elements of the preceding group, and the successive steps of ionization follow each other in rather quick succession.*

First of all, let us consider the relative intensities of the lines of alkalies and the alkaline earths in the sun and the spot spectra.

Alkalies.				Alkaline Earths.			
I.P. Volts.		Intensity.		I.P. Volts.		Intensity.	
		Sun.	Spot.			Sun.	Spot.
Na	5·11	30	60	Mg	7·65	30	30 *
K	4·32	6	15	Ca	6·08	20	25
R	4·16	—	1	Sr	5·67	1	3
Cs	3·81	--	?	Ba	5·12	—	—

* Russell, *loc. cit.* p. 130 ; the intensity given against Mg is that due to the $2p$–$3s$ line ; the resonance line of Mg, $\lambda = 2852$, is beyond the range.

The table shows that Ba is at least as highly ionized as Rb, though $I_1 - I_2 = ·96$ volt. Sr is only slightly less ionized than Rb, both in the sun and the spot. Calcium is less ionized than potassium. When we compare the intensity of the lines of Ca and Na, we find that in the sun they are almost equally ionized (calcium a bit more), but in the spot the recombination between Ca^+ and (e) is much less marked than between Na^+ and e.

Prof. Russell suggests that if the effect of radiation could be taken into account, the theory would be more improved, and the discrepancies could be explained. (See Russell, The Astro. Journ., May 1922.)

II.

It cannot be denied that the theory is always to be regarded as incomplete until the effect of the general field of radiation can be taken into account. But it is doubtful if this alone will explain all the discrepancies. Another factor, the consideration of which is presented below, seems to play a rather important part.

It is now well accepted that elements of the first group, Li, Na. K, Rb, Cs, have only one electron in the outermost ring, while metals of the second group, Mg, Ca, Ba, Sr, have two electrons in the outer ring. Besides, these two electrons are equally situated—in other words, whenever a Ca atom is subjected to the action of any physical agency tending to tear off the electrons, it will act equally on both of the valency electrons. In the case of the alkalies, it will act on

one electron * only. Let us take the case of electrolytic solutions. Here the electrical forces act *equally strongly from all directions*, and a calcium atom loses both electrons. The inner electrons being more solidly fixed to the nucleus do not get detached. A sodium atom, on the other hand, loses under the same conditions only one electron, because there is only one electron which can be torn off by the same agency from a Na-atom.

This proves that in the normal case both the valency electrons in Ca occupy nearly identical positions in the atomic system—they are contained in the same part of the atomic volume, and are fixed to the system with forces which are either identical or very nearly identical.

Let us now consider what will take place when a Na-atom and a Ca-atom are subjected to the same ionizing agencies, say bombarding electrons, light pulses, or thermal collisions. For the sake of simplicity we consider the first case only. We shoot at a Ca- and at a Na-atom with the same number of electrons, which are possessed of such energy that they can tear off just one outer electron when it hits at the right place in the atom. Now, assuming the atomic volumes to be the same and the I.P. to be the same, it is clear that the number of successful hits on a Ca-atom will be twice as great as the number of successful hits on a Na-atom, for in the outer volume calcium has two electrons, while sodium has only one. In other words, for the same strength of the ionizing agent, Ca-gas will be, roughly speaking, twice as highly ionized as Na-gas.

These considerations may be extended to all cases where ionization takes place by encounter, either with a light pulse or another atom.

In support of this view, an interesting observation by Millikan † may be cited here. Helium has two electrons which, according to Bohr and Lande ‡, are both in the same part of the atomic volume. Millikan finds that when helium gas is bombarded by α-particles, then, in one case out of seven, both electrons are simultaneously carried off by the α-particle. This could not take place if one electron was much nearer the nucleus than the other, and was attached to the nucleus with a greater force. If they are contained in the same part of the atomic volume, then, according to the laws of probability, in one case out of eight both electrons would

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* The strength of the ionizing agent is assumed to be not so large as to be able to tear off any one of the inner electrons.

† Millikan and Wilkins, Phys. Rev., March 1922.

‡ *Zeitschrift für Physik*, Bd. ix. p. 33.

come simultaneously in the same octant, and both would be carried off by an α-particle which chanced to pass close to them.

The alkaline earths resemble helium in so far that they have two valency electrons in the same part of the atomic volume in the outermost region of the atom *.

In solutions we have always Ca^{++}-atoms and never Ca^{+}-atoms, because the electric forces act equally strongly from all directions. In the cases considered by us, Ca^{+}-atoms are more probable than Ca^{++}-atoms, because the ionizing agencies act from one side only.

The cases of recombination of Ca^{+} and e, Na^{+} and e may be considered in the same light, and we find that in the case of Ca^{+} and e, recombination is more difficult than between Na^{+} and e. If we consider an Na^{+}-atom, we find that there are no electrons in the outermost ring, and the positive lines of force proceeding from the nucleus act equally strongly within the 4π solid angles about the atom. From whichever side the electron may approach the Na^{+}-atom, provided other things (energy, distance) are of the right order, the electron will be captured by Na^{+}. Not so in the case of Ca^{+}. It has still got a valency electron in the outermost rings; lines of force proceeding from the nucleus are strongly concentrated on it. In other words, to use the language of Stark, there is a negative patch on one side. An electron cannot be captured if it approaches Ca^{+} from this side. It can be captured only when it approaches the Ca^{+}-atom within only a definite fraction $\dfrac{4\pi}{n}$ of the total solid angle about the Ca-atom, where n is a number >1. We may call 'n' the "steric factor."

These considerations show that for an atom like Ca, ionization is easier, and recombination of Ca^{+} and e is more difficult than recombination of Na^{+} and e. For trivalent and tetravalent elements like Sc and Si these considerations will apply with even stronger force.

III.

It is a more difficult task to take account of the above facts in a statistical theory. To Boltzmann we owe the idea that when two atomic species A and B associate, every case of approach of A and B does not result in a combination, but only when A and B present to each other certain definite

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* It may be pointed out that Langmuir places helium at the head of the alkaline earths (see Loring, 'Atomic Theories').

parts, $\dfrac{1}{n_a}$, $\dfrac{1}{n_b}$, of their respective surfaces. Boltzmann was of opinion that for the formation of diatomic molecules ' n ' varies directly as the maximum valency of the element (viz. 2 for Ca, 7 for I, and so on) *.

In recent years the " steric factor " has been introduced into thermodynamics by Stern † in a new theory of the dissociation of I_2 vapour.

Stern considered the case from the standpoint of both thermodynamics and the kinetic theory, and came to the conclusion that ' n ' lay between 6 and 7 in case of combination of two I-atoms to form an I_2-molecule, thus lending colour to Boltzmann's belief. The kinetic theory is not very convincing, for the following reasons. According to dynamical principles, two particles A and B approaching each other from infinity cannot form a closed system until and unless they lose a certain fraction of their energy, presumably by radiation. Similarly, a molecule AB cannot be dissociated into A and B if the system does not absorb energy from the outside.

Thus a complete theory of ionization is incomplete without a consideration of the mutual action between radiation and matter, and we are beset with the same difficulties which have confronted all investigators on the subject since the days of Boltzmann.

Proceeding to the thermodynamical theory, the fundamental equation was derived from the equation

$$S_a + S_b - S_{ab} = U/T, \quad . \quad . \quad . \quad . \quad . \quad \textbf{(A)}$$

where $U = $ heat evolved, S_a, S_b, S_{ab} were calculated from the quantum theory involving certain assumptions. (Here ' a ' is Ca^+, b is ' e,' S_{ab} is Ca.) The above equations are derived on the assumption that the steric factor $n = 1$. Taking the " steric factor " into account, the probability that a and b would simultaneously present the definite portions, $\dfrac{1}{n_a}$, $\dfrac{1}{n_b}$, of their surfaces to each other is given by

$$W = \left(\frac{1}{n_a} \cdot \frac{1}{n_b} \right)^{N},$$

where $N = $ total number of particles of each species.

* Boltzmann, *Gastheorie*, Band ii. pp. 175–177. Jeans, ' The Dynamical Theory of Gases,' pages 209 to 217, 2nd edition.
† Stern, *Ann. der Physik*, vol. xliv.

The diminution in entropy

$$S_0 = k ln W = - R ln(n_a n_b),$$

so that instead of equation (A) we shall have

$$S_a + S_b - S_{ab} - S_0 = \frac{U}{T} ;$$

hence the equation of ionization takes the form, assuming that only one species of atom is present,

$$\log \frac{x^2}{1-x^2} P = - \frac{U_1}{2 \cdot 3 RT} + \frac{5}{2} \log T - 6 \cdot 5 + \log (n_a n_b).$$

The *effective ionization potential* I_e' now becomes

$$U - 2 \cdot 3 RT \log (n_a n_b)$$

or $$I_e = I - \frac{2 \cdot 3 RT \log (n_a n_b)}{23000} \text{ volts.}$$

For the electron and the alkalies we can take $n_b = 1$. For alkaline earths, if we follow Boltzmann, $n_a = 2$, but this evidently does not suffice in the present case. It may be pointed out that the present case is entirely different from that considered by Boltzmann, for we are considering the combination between an ionized atom and an electron, whereas Boltzmann considered the combination of two atoms. There is no reason why the steric factor should have the same value in both cases.

IV.

On the basis of the above formula, let us consider the effective I.P. of helium and the alkaline earths at different temperatures. Taking $n = 2, 4, 6, 8$ respectively, we have for

$$\left. \begin{array}{l} n = 2, \quad I_e = I - \cdot 060 \, m, \\ = 4, \quad I_e = I - \cdot 119 \, m, \\ = 6, \quad I_e = I - \cdot 158 \, m, \\ = 8, \quad I_e = I - \cdot 180 \, m, \end{array} \right\} \begin{array}{l} \text{the temperature being} \\ m\text{-thousands.} \end{array}$$

Russell is inclined to take the temperature of the spot = 4000° K. and that of photospheric emission = 6000° K. If this view be correct, the temperature of the spot is only slightly above that of the arc. But we find that in the arc the Ba line $\lambda = 5535$ is quite strong, while it is entirely absent from the spot. The discrepancy can, of course, be explained by assuming that the temperature of the arc is not uniform : the absorption of $\lambda = 5535$ is due to the cooler mantle of gaseous barium next to the air. But more extended research is required to test this point.

Taking the temperature of the spot and the sun to be 5000° K. and 7000° K. respectively, and $n = 8$, we find that for Ba, $I_e = 4.22$ and 3.86 respectively, *i. e.* in the sun barium is ionized like Cs, in the spot like Rb. This very nearly explains the complete ionization of Ba in the sun, as well as in the spot.

For Ca, $I_e = 5.18$ and 4.82 respectively. Thus in the sun, calcium is more ionized than sodium, while in the spot it ought to be a bit less ionized than sodium. This satisfactorily explains the behaviour of the Ca-lines in the sun and spot.

If we turn to the stellar spectra, we find that calcium 'g' disappears from the B8-stage. By using the original equation of ionization it was found that the temperature could not be less than 13000° K. But, according to Wilsing and Scheiner's intensity measurements, the temperature is only slightly above 10000° K. Taking $n = 8$, the effective I.P. becomes 4.21 volts; the ionization is complete at 10000° K*. Thus the introduction of the "steric factor" seems to bring down the temperature of different spectral classes in a line with the temperatures obtained from intensity measurements.

In my original calculation of the ionization of helium, the I.P. was taken to be 20.4 volts. This is now known to be wrong, for numerous workers have established definitely that real ionization begins at 25.4 volts. The former figures are therefore to be revised. Taking the steric factor $= 8$, the effective I.P. at 25000° K. becomes 21 volts, and ionization is 74 per cent. under concentrations corresponding to one atmosphere pressure. If $P = 10^{-1}$ atm., the ionization becomes complete. Thus the temperature of the Oa stars given in my former paper (Proc. Roy. Soc. Lond., May 1921, p. 151), remains unaltered.

Manganese.—Let us next consider the element Manganese, because the constitution of its series spectrum has been recently elucidated by Catalan (Phil. Trans. vol. 223). He finds the 1S term of Mn $= 59937$, the I.P. $= 7.41$ volts. The resonance lines of Mn are the triplet $\lambda = 4030.92$, 4033.21, 4034.62, so that, as far as the variation in intensity of its lines in stellar spectra is concerned, manganese is an ideal element. According to Lockyer, they occur as faint lines in the spectra of Ao-stars (intensity 1 on a scale 1–10, the Ca-line 4227 being of intensity 2). Thus, in spite of the fact that the I.P. of Mn is 1.23 volts higher than that of calcium, it is more highly ionized at T $= 10000$° K. than calcium. Manganese has got

* *Vide* M. N. Saha, "Elements in the Sun," Phil. Mag., Dec. 1920.

7 electrons in the outermost shell; its steric factor is there-fore expected to be much higher than that of calcium, which probably accounts for its comparatively high degree of ionization.

According to Lockyer, Mn^+ is represented in stellar spectra by $\lambda = 4344 \cdot 19$. It does not occur in King's furnace spectra *, though groups of lines due to Mn^+ occur about $\lambda = 3442–3497, 2914–2940$. $4344 \cdot 19$ is certainly not a resonance-line of Mn^+. Probably it is of the same type as $He^+ 4686$. According to Lockyer, this line vanishes from the stage ζ Tauri or β Persei. Thus in stellar ranges we have not only Mn and Mn^+, but Mn^{++} as well. None of the lines of Mn^{++} seems to be known.

Let us now treat some of the other elements in the order in which they occur in the periodic tables. For elements of the third group, *e. g.* Al, Sc, Y, La, neither the series classification nor the variation in intensity of lines in stellar spectra is satisfactorily known. We pass to the next group.

Group IV.   *Carbon, Silicon, Titanium.*

These elements are extremely interesting, because they have got 4-valency electrons, and the steric factor is expected to be unusually large not only for the neutral, but also for the singly- and doubly-charged atoms. As a result, the successive stages of ionization will follow each other in rather quick succession. Unfortunately, the knowledge of the spectra of these elements, as well as of the variation of their intensity in stellar classes, is not so well known.

Carbon.—The spectrum of this important element is one of the least known. The line $\lambda = 4267$ is supposed to be due to C^+; Lockyer, Baxandall, and Butler † treat the lines $4650 \cdot 8, 4647 \cdot 4$ as specially enhanced: we can assume that they are due to C^{++}. One wonders what lines are to be attributed to carbon itself. There are two strong lines, $\lambda = 6583 \cdot 0, 6577 \cdot 5$, which may be due to carbon. The variation in intensity of the lines of C^+ and C^{++} in stellar spectra is given below:—

	A.	B9.	B6.	B5.	B3.	B2.	B1.	B0.	O*e*5.	O*e*.
$\lambda = 4267 (C^+)$	—	—	2	3	5	5	3	2	—	—
$\lambda = 4650\cdot8 \atop 4647\cdot6 (C^{++})$.	—	—	—	—	1	4	10	25	15	$\oplus$

‡ Present, but intensity not exactly measured.

* King, Astro. Journal, vol. liii. (1921).

† Lockyer, Baxandall, and Butler, Proc. Roy. Soc. Lond. vol. lxxxii. p. 352.   Fowler, Report on Series in Line Spectra, p. 163.

The pair 4550·8 and 4647·6 are very prominent in the spectra of Novæ simultaneously with strong enhanced nitrogen lines 4634·34, 4640·82. If the above considerations be correct, carbon occurs in stellar spectra as C^+ and C^{++}, and the two stages follow each other in rather quick succession. This is to be expected of an element having a large "steric factor."

Silicon.—Lockyer [†] classified the lines of silicon in four groups, according to their mode of production, and has shown that each group is represented with the greatest intensity in stars at different stages of development. The following is compiled from the Harvard Annals:—

Stellar Class.	Group I. $\lambda = 3905$ (Si).	Group II. $\lambda = 4131$ 4128 (Si+).	Group III. $\lambda = 4359$ 4553 (Si++).	Group IV. $\lambda = 4089$ 5740
K	?	—	—	—
G	12 *	Absent.	—	—
F	Present.	Present.	—	—
A	—	Present.	—	—
B9.........	—	3	—	—
B6.........	—	3	—	—
B5.........	—	3	—	—
B3.........	—	2	1	—
B2.........	—	1	2	2
B1.........	—	—	4	5
B0.........	—	—	5	15
Oe5	—	—	?	12
Oe	—	—	?	6

* Intensity in the solar spectrum on Rowland's scale.

It is not quite certain if lines belonging to Group III. and Group IV. can be regarded as due to Si^{++} and Si^{+++} respectively. If Sommerfeld and Kossel's spectral displacement law [‡] be true, the spectrum of Si^{++} ought to have the same constitution as the spectrum of neutral magnesium, *i. e.* ought to consist of triplets and singlets. Group III. and Group IV. may be both regarded as due to Si^{++}, Group III. belonging to combinations like $1S-mp_2$, Group IV. to stronger combinations like $1S-mP$. This is of course only a suggestion. But there appears to be but little doubt that in the stellar range available for us, silicon occurs not

[†] The Si+-pair is present in the star γ Cygni (F8A) and is as strong as in *a* Cygni (A2F) (Lockyer, Month. Not. R. A. S. 1921). In F5A stars they are weakened, and entirely absent from the solar spectrum. Additional lines of Group IV. at $\lambda = 5740·2$, 4829·4, 4820·1, 4813·7 have been identified by Fowler in the spectra of β Crucis type B₁, Monthly Notices, vol. lxxvi. p. 196.

[‡] Sommerfeld, *Atombau.*

only as a neutral and a singly-ionized atom, but also at least as a doubly-ionized element, and the stages follow each other in rather quick succession.

Titanium.—The spectrum of Ti is very well studied, though the series classification is not yet known. In their general behaviour the lines of Ti, Ti^+, and Ti^{++} resemble the corresponding lines of Si, Si^+, Si^{++}, but titanium becomes ionized at a much earlier stage than Si, which is in accordance with the general rule that for elements belonging to the same group, heavier elements have got the smaller I.P. Lines of Ti^+ ($\lambda = 3759 \cdot 47$, $3761 \cdot 47$, $4578 \cdot 19$) are quite strong in the solar spectrum, and according to W. J. S. Lockyer *, they are more intense in F5G stars (ϕ Cassiopeiæ) than in A2F stars (α Cygni). The ionization of Ti commences much earlier than that of Si.

Group V. *Nitrogen, Phosphorus, Vanadium.*

The enhanced lines in the spectrum of Nitrogen and their occurrence in stellar spectra were first noted by Lockyer, Baxandall, and Butler †. The subject has been subsequently treated by Lunt, Fowler, and Wright.

According to these workers, the spectral lines of nitrogen can be divided into 2 or 3 groups according to the stimulus necessary for exciting them. The chief lines of Group I. are $\lambda = 3995 \cdot 15$, $4447 \cdot 20$, $4630 \cdot 73$, and the chief lines of Group II. are $\lambda = 4097 \cdot 48$, $4103 \cdot 54$, $4640 \cdot 82$, $4634 \cdot 34$. There is, besides, another line at $4379 \cdot 26$ which seems to belong to an enhanced group of still more pronounced type. Since lines of Group I. do not occur in the low-temperature spectral classes like G, F, and even at A0‡, I am inclined to think that they are due to N^+, or belong to some remote combination of the neutral nitrogen series. In the former case, Group II. would correspond to N^{++}. They first come out in the B2 classes (intensity 1), and gradually increase in intensity as we go to the still hotter stars, as the following shows :—

	29 Canis Majoris (Oe).	Can. Maj. (Oe5).	ε Orionis (B1).	β Centauri.	γ Orionis.
N^{++} (4097·45)...	18	6	4	2	1

* W. J. S. Lockyer, *loc. cit.*

† Lockyer and others, Proc. Roy. Soc. Lond. A. vol. clxxxii. p. 532. Lunt, Monthly Notices, lxxx. p. 534 (1920). Fowler, Monthly Notices, lxxx. p. 693.

‡ Of this last I am not quite sure.

The lines of N^{++} are very prominent in the spectrum of Novæ in their later stages simultaneously with He^+ 4686. and C^{++} 4650·8, 4647·6.

Nitrogen, having 5 electrons in the outer shell, would have a large " steric factor " for not only the neutral atom, but also for N^+ and N^{++}, which probably accounts for the quick succession of the different stages of ionization.

According to Lockyer [*], Protovanadium or V^+, as represented by the line $\lambda = 4053\cdot9$, is strongly developed in F5A stars, but shows a weakening in Ao-stars, and disappears somewhere about the B8-stage. In the stellar ranges we have, therefore, V, V^-, and V^{+-} in rather quick succession.

It will be seen that the above discussion mainly centres round the life-work of the late Sir Norman Lockyer. To him is due not only the idea, but also extended and elaborate studies of the enhanced and super-enhanced lines of elements, and their application to the study of the ordered sequence in stellar spectra. In this connexion, attention may be drawn to the views which he presented in his ' Inorganic Evolution of Elements.' He tried to impress the idea that the enhanced lines are due to some proto-form or fractional part of chemical atoms. But in those days the atom was an " elastic solid sphere," and his ideas did not find many adherents. The real significance of his works is being realized in these days of the Rutherford-Bohr theory of atomic constitution.

The present paper probably brings out the great importance of these studies to problems of atomic physics. Lockyer's studies have been continued by American and English workers, but the data hitherto available are not sufficient for the purpose. The above discussion, though scrappy, probably points out the direction in which these studies ought to be extended and amplified.

Note added during correction.—Since the paper was written, Russell has published a paper in the Astro. Journal, May 1922, where he has modified the view that Ba is absent from the sun. The 1S–3P line of Ba ($\lambda = 3071\cdot59$) has been identified with a line of intensity 00, the resonance line $\lambda = 5535$ being probably masked by an iron-line at $\lambda = 5535\cdot68$.

* Proc. Roy. Soc. Lond. vol. lxiv. p. 396; Phil. Trans. " On the Spectrum of *a* Cygnus," 1903.

CV. *The Ionization of Abnormal Helium Atoms by Low-Voltage Electronic Bombardment.* By FRANK HORTON, *Sc.D., and* ANN CATHERINE DAVIES, *D.Sc.* *

IN a recent paper in the Astrophysical Journal † F. M. Kannenstine describes experiments which show that with alternating electromotive forces, an arc in helium can be maintained, and even made to strike, at about 5 volts, provided the frequency of the alternations exceeds a certain limiting value. Evidence of the maintenance of arcs in helium at voltages below the first critical electron energy for this gas (20·4 volts) has been given by other observers ‡, and it might be expected that the limiting voltage for the maintenance of the arc in helium would be the difference between the normal ionizing voltage (25·2 volts) and one of the two critical electron energies for the production of radiation (20·4 volts and 21·2 volts), *i. e.*, the limiting voltage might be expected to be either 4·8 volts or 4·0 volts. Kannenstine's experiments seem to be the first in which a limit approximating to either of these values has been reached, and also the first in which the arc has been made to strike below 20·4 volts in helium. This striking of the arc at voltages below the resonance value in experiments with alternating electromotive forces is not, however, so much at variance with theory and with the results obtained by other experimenters as it appears to be at first sight, for what Kannenstine found was that after the cycle of voltages had once been completed, the arc struck at about 5 volts during subsequent cycles if the frequency of the alternations exceeded 220 per second. Hence the effect obtained was not the striking of the arc in normal helium, but the striking of the arc in helium containing some abnormal atoms for the production of which a higher voltage had been employed.

Kannenstine has pointed out that his results can be interpreted on the view that there is one of the states of the helium atom, intermediate to the normal state and the singly ionized state, in which the atom can exist for a perceptible interval of time, which he gives as about ·0024 sec. This

* Communicated by the Authors.
† F. M. Kannenstine, Astro. Phys. Journ. vol. lv. p. 345 (1922).
‡ K. T. Compton, E. G. Lilly, and P. S. Olmstead, Phys. Rev. vol. xvi. p. 282 (1920). A. C. Davies, Proc. Roy. Soc., A., vol. 100. p. 599 (1922).

particular state of the helium atom is referred to by Kannen-
stine as the " metastable " state, and his results are consistent
with the view that the particular state of the helium atom
which is metastable is the one into which the atom is thrown
as the result of an encounter with an electron having 20·4
volts energy. It will be shown later, however, that the
results obtained by Kannenstine can be explained without
the necessity of supposing one of the abnormal states of the
helium atom to be endowed with any greater stability than
other abnormal states.

The suggestion that a metastable state of the helium atom
exists was first made, on theoretical grounds, by Franck and
Reiche *, and the results of certain experiments made by
Franck and Knipping were interpreted by these experi-
menters as confirming the suggestion †. There is, however,
an essential difference between the term " metastable " as
used by Kannenstine and as used by Franck, Reiche, and
Knipping ; for the former uses the term in the sense that the
helium atom remains in that particular condition for a small,
but measurable, interval of time, whereas Franck and his
collaborators use the term to denote inability of the helium
atom to revert from that particular state to the normal state
without the assistance of external agencies.

Evidence on this point is provided by the results of
certain experiments performed by the authors, and described
by them in the Philosophical Magazine for November
1921. The conclusions to which these experiments led,
while agreeing with those of Franck and Knipping in some
respects, differed from them in regard to the production
of radiation in pure helium bombarded by 20·4 volts elec-
trons ; for while Franck and Knipping concluded that
radiation is not produced at all in these circumstances, the
authors concluded that an emission of radiation undoubtedly
takes place, but to a smaller extent than at 21·2 volts, the
second critical electron energy in helium. The results
obtained by the authors in the experiments referred to,
indicated that the abnormal atoms produced by 20·4 volts
electronic-atomic encounters reverted to the normal con-
dition less readily than the abnormal atoms produced as a
result of 21·2 volts encounters.

In the paper already referred to, the authors describe
experiments in which they attempted to detect ionization

* J. Franck and O. Reiche, *Zeits. f. Phys.* vol. i. p. 154 (1920).
† J. Franck and P. Knipping, *Zeits. f. Phys.* vol. i. p. 320 (1920).

when helium atoms which were in an abnormal condition were bombarded by electrons having energy in excess of 4·8 volts but less than 20·4 volts, and evidence is given which shows that ionization can be produced below 20·4 volts in this way. The attempts to demonstrate that such ionization could be produced by electrons having only 4·8 volts energy, were, however, unsuccessful. More recently further attempts to demonstrate this point satisfactorily have been made by the authors, and although the further experiments have, in some cases, only served to bring to light the existence of complicating factors which tend to frustrate the detection of the effect sought for, a few of the experiments which were made under conditions which varied over a wide range have yielded results which can be considered as satisfactory evidence of the production of ionization from abnormal atoms by bombardment with electrons having only about 5 volts energy.

The apparatus used in the present investigation was the same as that employed in the earlier experiments, and is described in detail in our paper. Its design allows of the production of abnormal atoms in the main tube by the action of radiation which enters from an auxiliary side tube. In this way abnormal atoms are obtained in the main tube without necessitating the presence of electrons in that tube. Radiation is produced in the side tube by bombarding the helium with an electron stream and passes into the main tube by the process of absorption and re-emission by other helium atoms, as shown in our earlier paper. Electrons and any positive ions produced in the side tube are prevented by the arrangement of electric fields from entering the main tube and so affecting the collecting electrode.

Having arranged for the presence of abnormal atoms in the main tube, the effects of bombarding them by a stream of electrons, the energy of which could be gradually increased, were investigated with each of the arrangements of electric fields designated in our earlier paper as those required for the obtaining of R, and (I−R), curves, *i. e*., curves showing the effects of radiation only, and curves in which the photoelectric effect of radiation opposes the ionization current. Hence, provided that in the absence of abnormal atoms the variation in the measured current with variation of the bombarding electron energy could be neglected, the production of ionization by bombardment of abnormal atoms at any given stage, would be indicated by a decrease of negative current at that stage, in an R arrangement, and by either a

decrease of negative current, or an increase of positive current, if an (I–R) arrangement were employed.

The cases in which evidence was obtained of the production of ionization of abnormal helium atoms by bombardment with electrons having about 5 volts energy are illustrated by the curve given below, in which the current measured by the electrometer is plotted against the energy of the electrons in the bombarding stream. In obtaining the results represented in this curve it was arranged to have a large quantity of resonance radiation coming into the main tube of the apparatus from the auxiliary side tube, and in order that an intense stream of electrons should bombard the abnormal
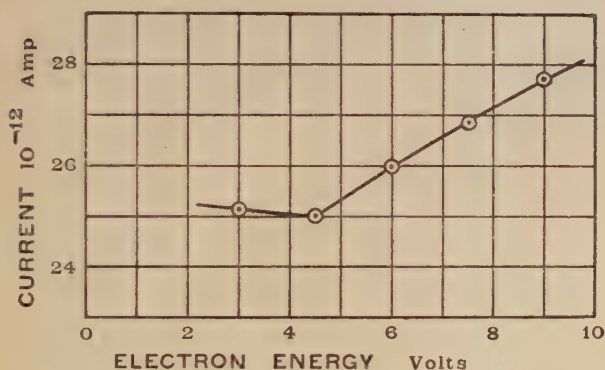
helium atoms formed in the main tube by this radiation, the filament supplying the bombarding electrons was heated to the limit of safety. Pure helium gas was constantly streaming through the apparatus during the experiment at an average pressure of 0·12 mm. The arrangement of electric fields was that required for an (I–R) curve. The curve shows that a positive current was measured which began to increase when the electron energy exceeded about 4·5 volts. The detection of a positive current below 4·5 volts is to be attributed to the ionization of 20·4 volts abnormal atoms by 21·2 volts radiation. The measured current at this stage could be made positive or negative by adjustment of the electric field in the V_2 space*. One of the complicating factors in the investigation was the recombination which occurred between the bombarding electrons and the positive

* See Phil. Mag. vol. xlii. p. 746 (1921).

ions which were produced by the radiation, for as the energy
of the bombarding electrons was increased, the distance
through which they travelled also increased, and hence the
amount of recombination became greater, and fewer positive
ions were collected. This variation in the amount of re-
combination occurring tended to mask the effect of ioni-
zation by the bombardment of abnormal atoms, and was
responsible for the negative results obtained in some instances.
The downward slope of the curve given in the figure, before
4·5 volts is reached, illustrates the reduction in the positive
current due to increased recombination as the energy of the
electrons is increased.

Some idea of the amount of recombination occurring
between the bombarding stream of electrons and positive
ions produced by the radiation which comes from the
auxiliary tube, is given by the following experiment :— With
the electric fields arranged for an (I–R) curve, and with V_2
adjusted so that the measured current was of the negative
sign, the current measured in a certain experiment was
$3·94 \times 10^{-13}$ amp. when the energy of the bombarding elec-
trons was 3·5 volts. By means of a current sent through a
coil of many turns of wire wrapped round the main tube, a
magnetic field could be applied parallel to the axis of this tube
This magnetic field served to prevent the electron stream
emitted from the hot filament from spreading laterally by
concentrating it into the central part of the tube, thereby
diminishing the possibility of encounters between these
electrons and positive ions produced throughout the space by
the radiation. It was found that the effect of switching on
the magnetic field (all the electric fields remaining as before)
was to cause a positive current of $0·53 \times 10^{-13}$ amp. to be
measured instead of the negative current of $3·94 \times 10^{-13}$ amp.
previously observed, the increase of positive current being
due to the smaller amount of recombination occurring.

Owing to the fact that when curves such as that given
in the figure were obtained the filament supplying the
bombarding electrons was extremely hot, it was desirable to
complete a series of observations as quickly as possible.
For this reason observations were taken at rather large
voltage intervals, and the genuineness of the observed effects
was tested by taking observations at decreasing values of the
voltage, as well as at increasing values, to see if the curve
retraced its course. The results do not make it possible to
decide with any great accuracy the minimum energy of the
bombarding electrons for which ionization is produced, but

they suffice to show that it lies between 4 and 5 volts. The results of these particular experiments do not, therefore, indicate whether the abnormal atoms which are being ionized are those resulting from 20·4 volts electronic-atomic encounters, or those resulting from 21·2 volts encounters.

It seems possible to account for the beginning of ionization between 4 and 5 volts without assuming that any one abnormal state of the helium atom is more stable than any other abnormal state. Although helium radiations corresponding to various voltages are present in the side tube, very little radiation corresponding to voltages other than 20·4 volts and 21·2 volts is likely to be passed into the main tube for the following reasons :—

α. Only such radiations as correspond to transitions from an abnormal to the *normal* state of the atom are capable of being absorbed by normal atoms and subsequently re-emitted.

β. Abnormal atoms which have absorbed radiation corresponding to voltages higher than 21·2 can revert to the normal state in several ways, the number of which increases as the energy of the abnormal atoms increases. It is therefore improbable that more than a small proportion of the abnormal atoms corresponding to higher voltages than 21·2 revert to the normal condition without passing through intermediate stages, in which case the only portion of the emitted energy which could be absorbed by normal helium atoms would be that corresponding to the final transition.

Thus the fact that ionization of abnormal atoms did not begin before about 4 volts in our experiments may have been due to the absence of abnormal atoms other than those corresponding to 20·4 volts and 21·2 volts, and the fact that ionization did begin between 4 and 5 volts, although consistent with the view that the 20·4 volts abnormal atom is "metastable" in the sense used by Kannenstine, is not sufficient to prove that this abnormal state is more "metastable" than any other abnormal state.

When arcs are maintained below 20·4 volts in helium, ionization of abnormal atoms only can be occurring. For the maintenance of low-voltage arcs, therefore, it is of importance that none of these abnormal atoms should be lost. When recombination takes place the energy of the radiation emitted during the final transition can be absorbed by normal atoms, and the minimum energy which must be supplied in order that every type of abnormal atom thus produced may be ionized, is 4·8 volts. If the bombarding

electron-energy has a smaller value than this, some of the abnormal atoms produced by absorption of radiation corresponding to the final transitions at recombination cannot be re-ionized, and hence the arc can no longer be maintained.

In the experiments of Kannenstine, the re-striking of the arc below 20·4 volts must have been due to the fact that the interval which elapsed between the breaking of the arc, and the re-application of the energy necessary for the ionization of abnormal atoms, was sufficiently short for there to be still a considerable number of abnormal atoms present at the end of the interval. This does not necessarily imply that any particular abnormal atom remained in the abnormal condition for the whole of this time, for the presence of abnormal atoms at the end of the interval would be secured if the radiation were absorbed and re-emitted by other atoms several times during the interval. It does not follow, therefore, that the period of ·0024 second measured by Kannenstine represents the " life " of individual abnormal atoms.

In the event of radiation being absorbed and re-emitted several times during the interval, there would be, as we have already shown, a marked falling off in the number of abnormal atoms corresponding to voltages greater than 21·2, so that the abnormal atoms present at the end of the interval would be almost entirely of the types corresponding to 20·4 volts and 21·2 volts, requiring between 4 and 5 volts energy for their ionization. It thus appears that the maintaining of the arc down to 4 or 5 volts and not below this value, and the fact that it re-strikes when this voltage is again established, can be accounted for without assuming the metastability of any abnormal state of the helium atom.

CVI. *The Ionizing Potential of Positive Ions.*

To the Editors of the Philosophical Magazine.

GENTLEMEN,—

IN a paper on impact ionization by low-speed positive H ions in hydrogen, published in the November number of the Philosophical Magazine, p. 806, Mr. A. J. Saxton gives a general account of the results that have been obtained on the subject of ionization by positive ions.

As an example of some of the experiments which I made, Mr. Saxton refers to one where the ionization is produced by electrons and positive ions moving in hydrogen at 8 millimetres pressure under a force of 700 volts per centimetre ('Electricity in Gases,' p. 317), and states that "the M.F.P. of a positive ion would be very small at this high pressure, so that it could not obtain an unimpeded fall through more than a fraction of a volt," and makes some suggestions as to how the ions were generated in these experiments. Mr. Saxton's conclusions appear to be based on the supposition that all free paths are nearly equal to the mean free path and that the number of molecules ionized by single collisions with positive ions is too small to be taken into consideration.

The experiment in question was published in the Philosophical Magazine, vi. p. 607. Nov. 1903, where I gave a method of estimating approximately the ionizing potential of positive ions from the determinations of the number of molecules ionized by a positive ion in moving through a centimetre of the gas in the direction of the electric force. The ionizing potential of positive ions in hydrogen was found to be 20 or 30 volts. The voltage depends on the length of the mean free path, and it was also stated in the original paper that a somewhat lower voltage would be obtained if the mean free path of a positive ion was the same as that given by Meyer for a molecule of the gas. For hydrogen at normal pressure and temperature the mean free path given by Meyer is $17 \cdot 8 \times 10^{-6}$ cm. ('Kinetic Theory of Gases,' p. 192).

The free paths were not known very accurately at that time, but as a result of recent investigations a more exact formula connecting the viscosity of a gas and the mean free path of a molecule has been obtained. It is therefore of interest to calculate the ionizing potential of positive ions in

hydrogen taking $11 \cdot 25 \times 10^{-6}$ cm. as the mean free path in the gas at normal pressure and temperature, which is the mean free path of a molecule of hydrogen given by Jeans in the last edition of his treatise on the Dynamical Theory of Gases.

On this hypothesis, the number of collisions made by a positive ion in traversing a centimetre of the gas at 8 mms. pressure is 936, and it was found by the experiment that on an average the number of molecules ionized by a positive ion in moving through one centimetre is ·059. For a first approximation the velocity of agitation of the ions may be neglected in comparison with the velocity in the direction of the electric force. Thus the ratio of the number of molecules ionized to the total number of collisions by positive ions is $1 : 16,000$.

If all the ionization by positive ions be attributed to collisions which terminate free paths greater than x, then $\epsilon^{x/l} = 16,000$ where l is the mean free path, so that $x = 9 \cdot 68 \times l = \cdot 0103$ cm., and the potential fall along these paths would be greater than $7 \cdot 2$ volts. The velocity corresponding to this voltage is much greater than the velocity of agitation of the molecules of the gas, so that the longer free paths of the positive ions would exceed the value ·0103 cm. by the factor $1 \cdot 4$, and the potential fall would also be greater in the same proportion. Hence the ionizing potential of positive hydrogen ions, as deduced from the above experiment, must be at least 10 volts.

In order to make a more accurate estimate of the ionizing potential it would be necessary to take into consideration the initial velocities of the ions in estimating the number of paths which are terminated by a velocity above a certain value V, so that the above calculation only gives a lower limit to the ionizing potential. If other experiments of the series be considered where the gas is at pressures lower than 8 millimetres, it will be found that the ionizing potential of the positive ions in hydrogen is above 15 volts.

Yours faithfully,

7th November, 1922.   J. S. Townsend.

CVII. *The Propagation of a Fan-shaped Group of Waves in a Dispersing Medium. By* G. BREIT, *National Research Fellow, U.S.A.*[*]

BY a *Fan-shaped Group* is meant a group of waves in which the successive waves are plane, and have a small constant inclination with respect to each other. It may be produced by reflecting a plane wave-train in a rotating mirror or by moving a source of light in the focal plane of a lens. A picture of such a group is given in fig. 1.

Fig. 1.
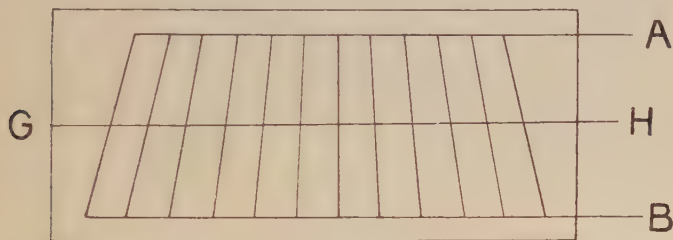
The propagation of a fan-shaped group in a dispersing medium has been discussed by Gibbs[†]. His calculation shows that at a given point of the group (such as a maximum of intensity) the orientation of the elementary waves is constant in spite of the fact that each of the waves rotates during its progress as a result of the dispersion. The important consequence of this result is that the image of a fan-shaped group, which is first passed through a dispersing medium and then through a lens, does not experience a displacement on account of the dispersing action of the medium.

Having recalled this result of Gibbs, Prof. Ehrenfest showed that the experiment recently proposed by Einstein [‡] is not capable of settling Einstein's question, for the latter incorrectly supposed that, according to the classical wave theory, the image of a fan-shaped group formed by a lens is displaced on account of the dispersing action of a medium which is put in front of the lens. [Another criticism leading to the same result has been published by Raman[§] ; and Einstein also revised his views [‖].]

In this connexion, it is of interest to point out that the result of Gibbs can be understood without calculation in the following manner :—

Statement of Result.

Consider the group drawn in fig. 1. The waves are more crowded at A than at B. The medium is thus disturbed at a higher frequency at A than at B, and the velocity or propagation is therefore different at A and B. In spite of this, as shown by Gibbs, the waves passing a point moving with the group velocity—appertaining to the frequency of the group at the point—pass that point always in the same orientation.

Thus it is required to show that at a given point of the group the orientation of the elementary waves is constant.

Proof in Special Case.

Consider the special group of fig. 2, obtained by superposing the sinusoidal waves of wave-lengths λ_1, λ_2 drawn on

Fig. 2.
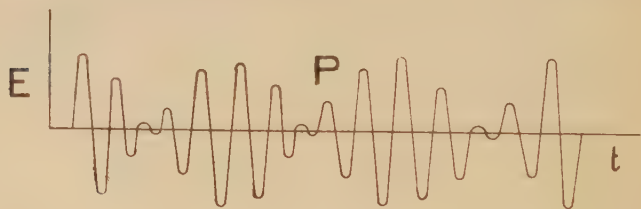
Fig. 3.
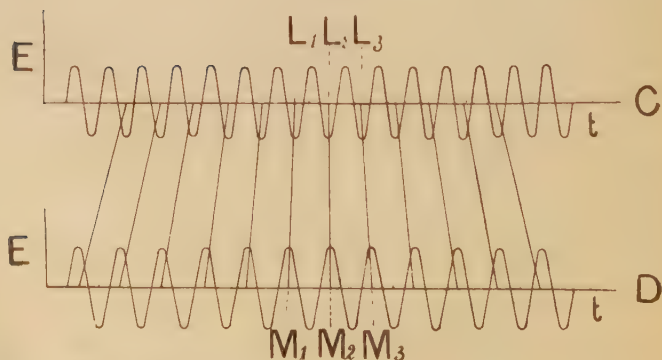
fig. 3. The minimum P is the point at which λ_1, λ_2 destroy each other. Hence the lines L_1M_1, L_2M_2 joining points of opposite phase of λ_1, λ_2 on fig. 3 are vertical at P. If CD is

made equal to A B of fig. 1. and if in fig. 1 the difference in λ on the top and on the bottom is just $\lambda_2 - \lambda_1$, then these lines represent the waves of fig. 1. Thus at P the direction of the elementary wave is always vertical, and therefore constant. If the difference in λ in fig. 1 is $\lambda'_2 - \lambda'_1 \neq \lambda_2 - \lambda_1$, the argument applies if we make $\dfrac{\text{C D}}{\lambda_2 - \lambda_1} = \dfrac{\text{A B}}{\lambda'_2 - \lambda'_1}$. In the general case the argument may be stated as follows :—

Proof in General Case.

The motion of a point of the group is such as to keep the phase-difference between two nearly equal wave-lengths constant to within quantities of the first order in $d\lambda$. Therefore if the angle between consecutive waves is so small that dispersion effects may be treated as small quantities of the first order in $d\lambda$, then the motion of a point of the group may also be said to be such as to keep constant the phase-difference between the wave-lengths λ'_1, λ'_2 at the top and at the bottom of an elementary wave.

Let now two points be considered both moving with the group velocity corresponding, say, to the bottom, and both situated in the surface of an elementary wave at a given instant—one at the top and the other at the bottom of the wave. The wave moves slower at the top than at the bottom, but both top and bottom move faster than the two points. Thus the two points are overtaken by waves coming from behind, and each of these waves is turning during the motion.

Since now the points move so as to keep the phase-difference between the wave-lengths at the top and at the bottom constant, and since the points have been once in the surface of a wave (so that the constant phase-difference is zero) it is apparent that if a wave reaches the top point it simultaneously reaches the bottom point, and thus the orientation of a wave is unchanged if it is picked out by a point moving with the group velocity [*].

Briefly : both motion with group velocity and motion with

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[*] It is essential to the argument here given that the words " group velocity " should have a definite meaning. This implies that the Fourier analysis of the group is confined to a sharp band on the scale of variations in wave-length along the aperture. A closer examination shows that this condition is fulfilled if the angle between the first and last waves is large in comparison with the least angle distinguishable through diffraction.

a constant orientation of the elementary wave are defined
by the same criterion, viz. zero phase-difference between two
adjacent wave-lengths. Thus it becomes obvious that one
implies the other.

To Prof. Lorentz and Prof. Ehrenfest the author is very
grateful for the discussion of the subject.

CVIII. *On the Flow of Liquids under Capillary Pressure.*
   By Eric Keightley Rideal [*].

THE rate of penetration of liquids into capillary porous
materials, of importance not only in biochemical pro-
blems but also in the study of the phenomena of adsorption
by materials such as charcoal and substances constituting
the membranes of semi-permeable osmometers, has attracted
but little attention. Bell and Cameron (Journ. Phys. Chem.
x. p. 659 (1906)) showed that in the case of a few liquids
the rate of movement of a liquid moving through a hori-
zontal capillary was such that the relationship $x^2 = kt$ (where
x was the distance traversed in time t) held within the limits
of accuracy of the experimental method. Cude and Hulett
(J. A. C. S. xlii. p. 391 (1920)), in their study of the rate of
penetration of charcoal by water, obtained for the initial
period of penetration a similar relationship. Washburn
(Phys. Rev. xiii. p. 273 (1921)) has examined the problem
in more detail, and deduced for the conditions of horizontal

flow the equation $x^2 = \dfrac{\gamma \cos . \theta}{2\eta} rt$, where γ is the surface

tension, η the viscosity, r the capillary tube radius, and θ the
angle of wetting. The validity of a similar expression was
tested experimentally for liquids moving through capillaries
under the influence of their own capillary force as well as a
constant large external pressure. For all liquids which wet
the tube wall of the material, $\cos \theta$ is evidently equal to unity.
The same value obtains for liquids which do not wet the tube,
since the angles of wetting noted in the literature are pro-
bably fictitious, and are due to observations on the alteration
in the radius of curvature at some point distant from the
point of contact with the tube wall, and not at the contact
point itself. Washburn has assumed that Poiseuille's law
holds true during the flow after the initial period of turbu-
lence has ceased, and has calculated with the aid of this

expression the rate of flow from the driving pressure made up of three separate pressures, the unbalanced atmospheric pressure, the hydrostatic pressure, and the capillary pressure. In the case of horizontal tubes the first two pressures are eliminated. The effective total driving pressure, however, varies with the length of the column, since the frictional resistance to the flow increases with the length of the column in the capillary tube. With this correction a somewhat different expression from that of Washburn is obtained for the rate of penetration, which, however, reduces to the form obtained by him on the neglect of terms which are insignificant except for very small and very large values of x the distance of penetration. A simple derivation of the relationship may be obtained in the following manner :—

Fig. 1.
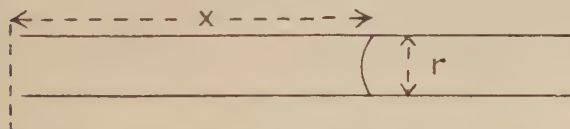
The forces acting on a column of liquid x cm. long in a capillary tube r cm. radius are :

(1) The surface tension forward, in magnitude $2\pi r\gamma$.

(2) A retarding force due to the viscosity of the liquid in the tube.

According to Poiseuille's law, neglecting the slip factors, this retarding force may be expressed in the following form,

$$\frac{dv}{dt} = P\frac{\pi r^4}{8\eta x},$$

where $\frac{dv}{dt}$ is the rate of flow : hence, solving for P the pressure, we obtain

$$P = \frac{8\eta x}{\pi r^4}\frac{dv}{dt} = \frac{8\eta x}{\pi r^4}(\pi r^2\dot{x}) = \frac{8\eta x \dot{x}}{r^2}.$$

The retarding force acting on the column is consequently

$$F = \pi r^2 P = 8\eta x\pi\dot{x}.$$

The net force acting on the column thus varies not only with the length of tube wetted, but also with the velocity of flow, and is equal to

$$2\pi r\gamma - 8\eta x\,\pi\dot{x}.$$

The mass of the column in motion is $\pi r^2 x \delta$, where δ is the density of the liquid ; hence

$$\pi r^2 x \delta \, \ddot{x} = 2\pi r \gamma - 8\eta x \, \pi \dot{x},$$

or $\qquad \ddot{x} = \dfrac{2\gamma}{\delta r x} - \dfrac{8\eta \dot{x}}{r^2 \delta},$

or $\qquad \ddot{x} + a\dot{x} = \dfrac{b}{x}.$

On integrating this expression, we obtain

$$c + t = \frac{1}{2}\frac{a}{b}x^2 - \frac{1}{a}\log x + \frac{1}{2}\frac{b}{a^3 x^2} + \frac{11}{4}\frac{b^2}{a^5 x^4} - \frac{17 b^3}{a^7 x^4},$$

or $\qquad t = \dfrac{2\eta}{\gamma r}x^2 - \dfrac{\delta r^2}{8\eta}\log x + \dfrac{\delta^2 r^5}{512\eta^3 x^2} + \cdots$

For small values of r this equation reduces to the form given by Washburn for a liquid wetting the tube wall or

$$x^2 = \frac{\gamma r}{2\eta}t.$$

Experiments on the rate of penetration of liquids moving through horizontal glass capillaries under their own capillary pressure alone without any external force, indicated that when the liquid wets the tube the angle of wetting is zero, and that the penetration coefficient is given by the expression

$$\sqrt{\frac{\gamma r}{2\eta}}.$$

Experimental.

A piece of capillary tube 1·2 metres long, of average internal diameter 0·708 mm., chosen for uniformity of bore, was mounted horizontally in a condenser tube maintained at 20° C. One end of the tube was fused into the side of a wide boiling-tube containing the liquid to be tested. In each experiment the boiling-tube was filled until the surface of the liquid just covered the mouth of the capillary. The liquid was drawn through the capillary by suction, and forced back to within 15 cm. of the end by compressed air so as to thoroughly wet the tube. This operation was performed several times, the final removal of the liquid being accomplished very slowly to ensure removal of all excess

liquid from the tube wall. The rate of flow was determined
by means of a stop-watch at increments of 10 cm. along the
tube. In the following curves (I.) are shown the distance
traversed in various times for a series of aliphatic alcohols;
and from these data the derived curves (II.) were obtained,
indicating the relationship between x and $\sqrt{t}$, which, ac-
cording to the theory outlined above, should be linear for
values of x within a relatively wide range.

Curves (I.).
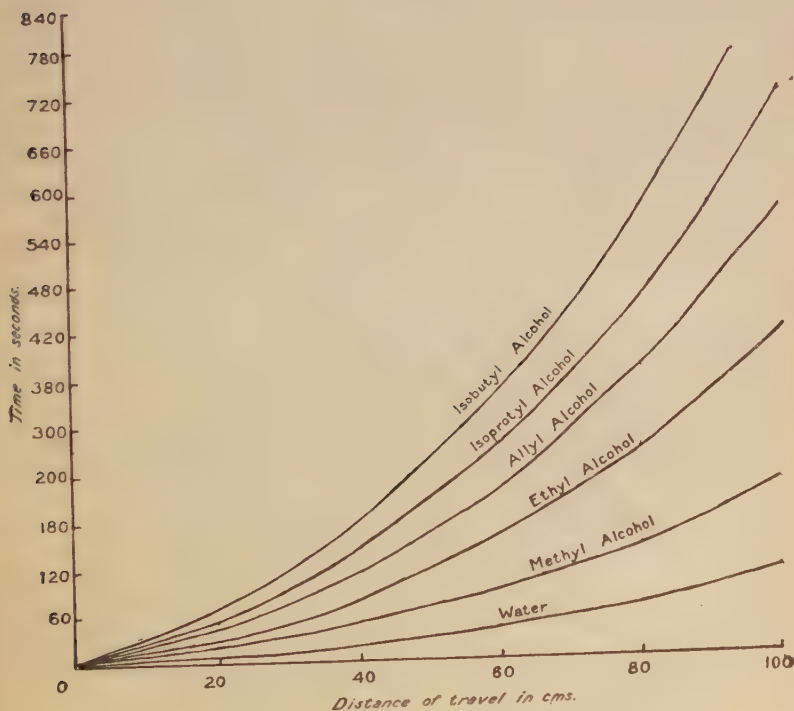
The values of the penetration coefficients obtained from
the slope of these derived curves are given, together with
those calculated from the data on the viscosity and surface
tension given in the tables of Landolt-Börnstein according

to the expression $K = \sqrt{\dfrac{\gamma r}{2\eta}}$, where $r = 0.0354$ cm.
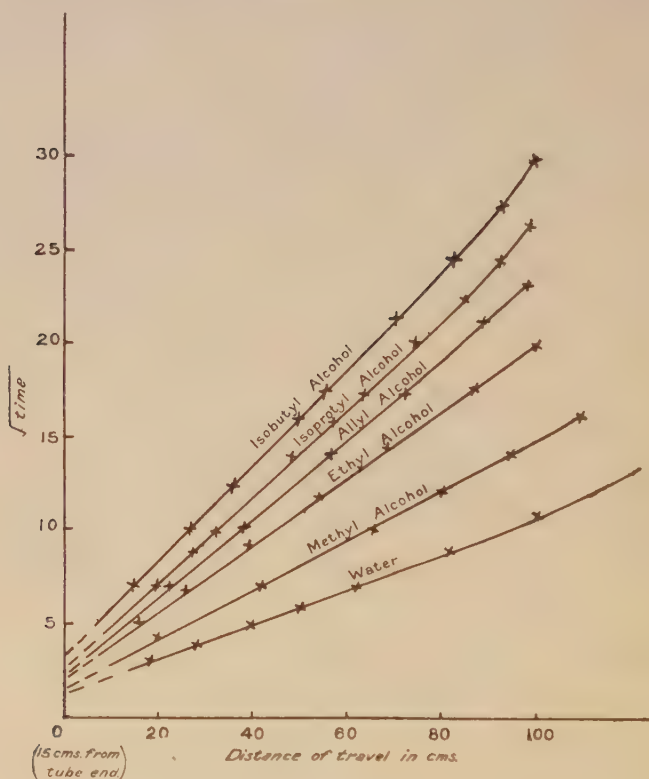
Substance.	K calc.	K. obs.
Isobutyl alcohol	3·75	3·70
Isopropyl ,, ...	4·10	4·20
Allyl ,, ...	5·43	4·82
Ethyl ,, ...	5·52	5·65
Methyl ,, ...	8·16	7·90
Chloroform	8·73	8·60
Benzene	8·90	9·90
Ether	11·38	10·95
Water	11·31	11·40
Acetone	11·70	12·70
Ethyl acetate	9·60	10·20

It will be noted that the agreement is remarkably close in
those cases for which the values for the surface tensions and

viscosities at 20° C. have been determined with accuracy, *e g.* water and chloroform. In the other cases, with the possible exception of benzene, the variations in the values of the determinations by various investigators for these quantities is sufficiently large to account for the discrepancies.

A few experiments were made on the rate of penetration of ethyl alcohol water mixtures, when the following values for the penetration coefficients were obtained :—

Per cent. Ethyl alcohol.	*K* obs.	*K* calc. static.
0	11·40	11·31
20	5·6	6·2
40	5·1	4·9
60	5·0	4·6
80	5·4	5·07
100	5·65	5·52

It will be noted that a minimum is obtained at ca. 50 per cent. in agreement with the calculated * values ; but the discrepancy between the calculated and the observed values is considerably greater than the experimental error. Data are lacking on the dynamic values of the surface tensions of alcohol-water mixtures ; but these figures, including some experiments on the rate of flow of dilute soap solutions, indicate that the surface film in the capillary tube is being continually renewed during its progress through the tube. The method is consequently applicable to the determination of dynamic surface tensions of mixtures which frequently differ considerably from the stated values.

The penetration coefficient of a liquid is a physical constant of importance in that it is related to similar constants for gases and solids. Determination of the Maxwellian period of molecular relaxation in gases (Jeans, ' Dynamical Theory of Gases,' p. 261 ; Boltzmann, *Vorlesung über Gas Theorie*, pt. 1, p. 167) or the " sensibilité " of Perrin (*Ann. de Phys.* xi. p. 21 (1919)) indicates that the molecules, even when acted upon by the mutually relatively feeble forces of adhesion in the gaseous state, are highly damped, the relaxation period for nitrogen at 0° C. and 760 mm. being $1·66 \; 10^{-12}$ sec. In the case of solids, the force fields or adhesion forces are naturally much greater, causing the molecular vibration to be even more highly damped. For

* Dunstan, J. C. S. lxxxv. p. 824 (1904); Firth, J. C. S. xxxiii. p. 268 (1920).

metals, the period of relaxation (Langmuir, Phys. Rev. viii. p. 171 (1916)) is in many cases identical with the period of electronic vibration as determined by the ultra-violet radiation frequency (10^{-14} sec.), whilst for non-metals a close approximation to the relaxation period is to be found in the vibration frequency of the residual rays or natural infra-red vibration frequency (10^{-12} sec.) (Rideal, Phil. Mag. xl. p. 462 (1920)). For liquids, it is to be anticipated that the period of molecular relaxation should lie between these values, *i. e.* ca. 10^{-12} sec., the more polar the medium the shorter being its time of molecular relaxation. This period is given by the expression $\tau = \dfrac{\eta}{P}$, where η is the viscosity of the fluid and P the pressure.

In the case of liquids, P is identical with the internal adhesional pressure $P_2 = \dfrac{\rho^2}{m} \int f(r)dr$. Although $f(r)$ cannot be evaluated without a knowledge of the nature of the intramolecular forces, yet it is possible to obtain values for P from various sources, such as the $\dfrac{\alpha}{v^2}$ term of van der Waals' equation, from the latent heat of evaporation (Stefan, *Wied. Ann.* xxix. p. 655 (1886)), from the coefficients of expansion and compressibility, or from the surface tension of the liquid. In this latter case

$$\frac{1}{\tau} = K \frac{\gamma}{\eta} = K'\,(\text{penetration coefficient})^2.$$

In the following tables are given the approximate values of P, being the mean values of the determinations by the various methods, and the values of τ calculated therefrom, compared with the values obtained from the penetration coefficient. [K_0 being evaluated from the data for ether.]

Substance.	P atmospheres.	$\tau\ 10^{12}$.	Penetration coefficient K.	$\tau\ 10^{12}$ from K.
Ether	1,590	1·31	11·38	(1·31)
Chloroform	2,200	2·13	8·93	2·10
Isobutyl alcohol	1,900	14·7	3·76	12·2
Isopropyl ,, ...	2,370	9·4	4·10	10·1
Acetone	2,520	1·32	11·03	1·41
Water	18,050	0·54	11·26	1·35

Better agreement is scarcely to be anticipated, since the values of P for any liquid are not known to any degree of accuracy; P for benzene, for example, varying from 1300 to 3810 atmospheres, according to the method employed for its evaluation (Hildebrand, J. A. C. S. xli. p. 1072 (1919)). The parallelism between the two sets of determinations and the decrease in the molecular relaxation period or increase in molecular " sensibilité " with increasing polarity is, however, clearly marked.

Summary.

The rate of penetration of a liquid into a fine capillary under its own forces is shown to be expressed by the relationship $t = \dfrac{2\eta}{\gamma r} . x^2 - \dfrac{\delta r^2}{8\eta} \log x$. For relatively large capillaries the penetration coefficient is $\sqrt{\dfrac{\gamma r}{2\eta}}$.

The experimental determination of the coefficient is shown to agree with the calculated values. In the case of mixed solvents the dynamic surface tensions and not the static values are probably the governing factors.

The reciprocal of the penetration coefficient is proportional to the square root of the period of molecular relaxation as defined by Maxwell, and on analogy with reactions in the solid state is probably important in reactions taking place in liquid media.

The writer is indebted to Mr. R. L. Huntingdon for assistance in the experiments detailed in this paper.

Chemical Department,
Cambridge University,
June 5th, 1922.

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CIX. *On a Balance Method of measuring X-Rays.* By Professor S. RUSS, *D.Sc.*, and L. H. CLARK, *M.Sc*, *Physics Department, Middlesex Hospital* *.

THE frequent and prolonged running of X-ray tubes calls for some convenient method of recording continuously the intensity of the X-radiation emitted during the period of excitation. The balance method described below indicates at any instant this intensity and is capable of giving a continuous record of it. It is suitable for the measurement of the ionization produced by a powerful beam of X-rays.

* Communicated by the Authors.

In principle, this balance method is similar to that devised by Rutherford and described by Bronson (Phil. Mag. vol. xi. p. 143, 1906). Electrical communication is made between two ionization chambers by joining the insulated electrode which each contains. Initially these electrodes are earthed, but the chambers are maintained throughout at a constant potential difference by connecting them to the terminals of a battery of cells. One chamber is exposed to a beam of X-rays and the other simultaneously to the radiation from a very small mass of radium. Under the action of the two consequent ionization currents, the potential of the insulated electrodes rises to a steady value when a balance is set up. The potential of the electrode is indicated by attaching to it a gold leaf, the deflexion of which is then used to indicate the intensity of the X-radiation.

Fig. 1 represents a working model on the lines indicated.

Fig. 1.

Vessel A supports within it a small ionization chamber C, measuring $0.4 \times 1.0 \times 1.5$ cubic cm., the two small opposite ends of which are closed by windows of aluminium foil. This chamber carries along its long axis an insulated brass electrode, connected by a fine wire to the gold-leaf support in vessel B. One wall of this vessel is made of very thin aluminium leaf to allow entry of α radiation from a small mass of radium.

Fig. 2.
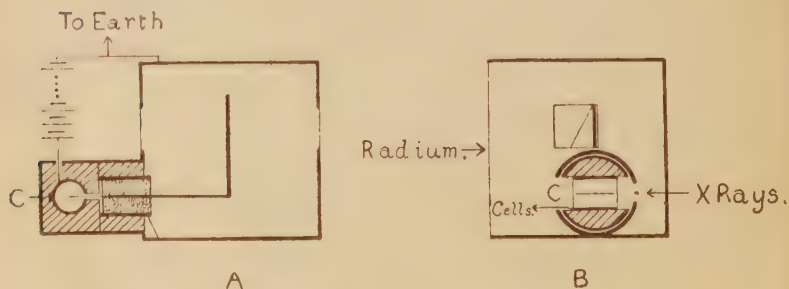
Another form of the apparatus is shown in fig. 2. In this case the ionization chamber C consists of a lead cylinder 1.2 cm. long and 0.8 cm. in diameter fitted with an axial

electrode. Each end of the chamber is closed by a very thin mica window coated with aluminium leaf. A lead diaphragm 0·5 cm. in diameter carrying a minute lead disk at its centre allows an annulus of radiation to enter the chamber C, and pursue a direct air path to the further window, without impinging upon either the interior of the chamber or the axial electrode.

Under working conditions 200 volts are applied to the chamber C, the connecting tube and the vessel B are earthed, the central electrode and gold leaf being initially earthed— then left insulated. The deflexion of the gold leaf is observed through a tele-microscope or the image of the leaf is projected on a screen.

From a practical point of view it is important to know how small a quantity of radium can serve to balance the ionization current due to a powerful beam of X-rays. In the two types of apparatus shown diagrammatically in figs. 1 and 2, 0·08 mgm. of radium bromide served to balance currents set up in the given ionization chambers C, when the latter were approximately 30 cm. from the anticathode of a Coolidge tube. The radium was spread over a circular area of 2 cm. diameter, the preparation being covered by a thin layer of mica, which allowed the α rays to pass through.

With the ionization chamber C set up at a definite distance from the anticathode of a Coolidge tube, a series of observations was made to determine the way in which the deflexion of the gold leaf varied with the intensity of the X-radiation from the bulb. Keeping the spark-gap constant, the filament current to the tube was varied over wide limits and the intensity of the X-radiation was measured by a distant electroscope. Simultaneously with these readings the balance deflexion was observed.

The curve of fig. 3 was obtained with the apparatus shown in fig. 2. It indicates a simple relationship between the balance deflexion and the intensity of the X-radiation entering the chamber C. The conditions of the experiment were as follows :— 180 volts on chamber C, distance of C from anticathode 27 cm., alternative spark about 4 inches : the distance of the radium from the aluminium window in B, 2·2 cm.

The time taken for a state of balance to be set up was found to be independent of the intensity of the X-ray beam. If the leaf be started from its zero position it moves more slowly to its position of balance with a weak than with a strong intensity, but the smaller actual displacement practically

compensates this so that the time for a balance does not vary
appreciably over a considerable range of intensity. This is

Fig. 3.
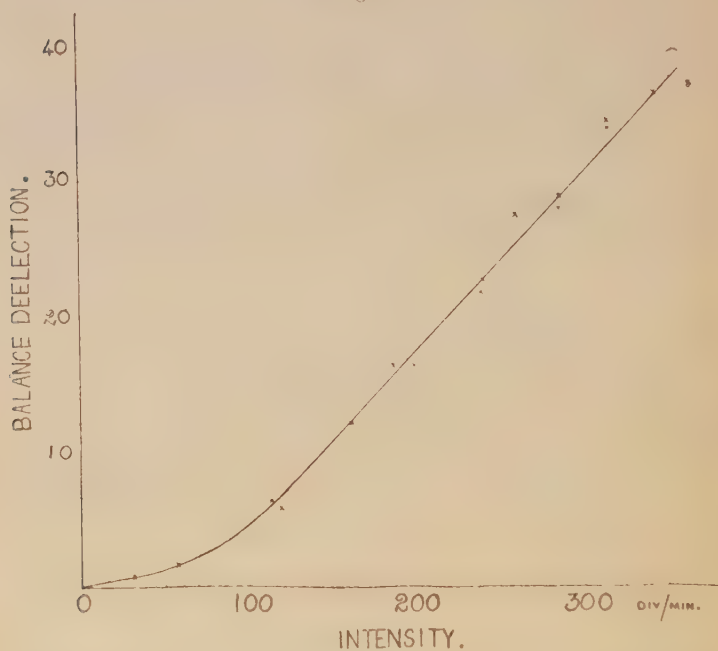
well shown by the curves of fig. 4, which were obtained with
the model of fig. 1. The position of the gold leaf was read
at 3-second intervals from the moment the excitation of the
Coolidge bulb was begun. The ultimate steady deflexions
were 21·2, 29 2, and 47·5 respectively, yet the time taken to
reach these values was approximately 30 seconds in each
case.

Saturation of the ionized air in C is of course essential if
the method is to be used for the quantitative measurement
of the intensity of the beam of X-rays. With 200 volts on
C, it was found that saturation was obtained provided that
the radium was in such a position that the leaf balanced at a
deflexion corresponding to a voltage on the leaf of about
120 ; this left 80 volts for the ionization current. If the
radium is removed to a greater distance the leaf deflects
more, but the corresponding rise in voltage now leaves
insufficient for saturation. One result of this is that a
spurious steadiness obtains in the balance : when saturation

occurs in C oscillations are easily observed, vide fig. 4. The oscillations under our experimental conditions were not due to fluctuations in the radiating power of the radium, for, if the ionization in C is produced by a radium source giving beta and gamma rays, a very steady balance is obtained. The oscillations which may be produced when an alpha ray

Fig. 4.

occurs in C oscillations are easily observed, vide fig. 4. The
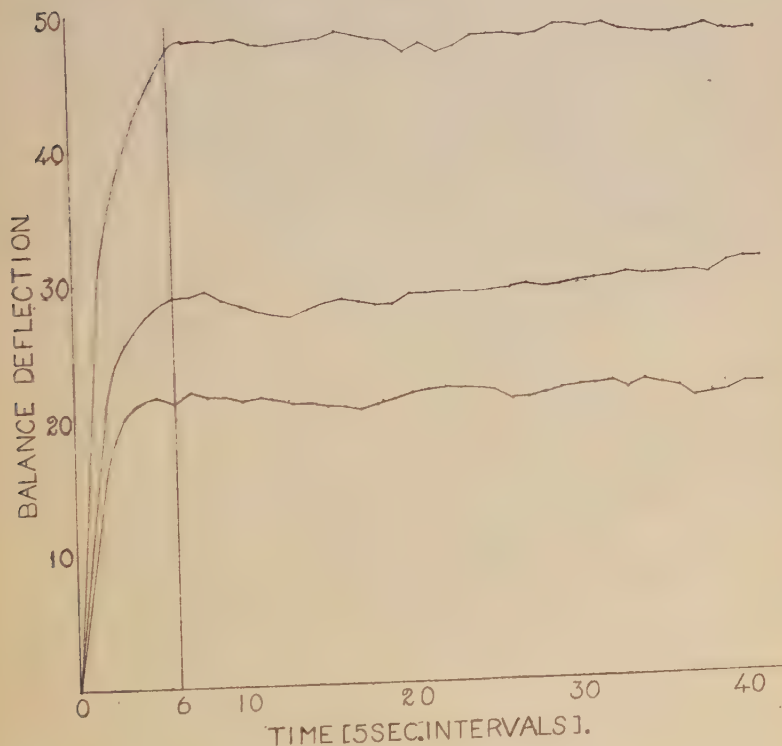
source is used has been studied by Geiger (Phil. Mag. vol. xv. p. 539, 1908), who showed that they were due to actual variation with time in the number of alpha particles emitted. He showed that for similar ionization by beta radiation, oscillations still occurred, though their significance was minimized owing to the greater numbers involved. It might be possible to detect oscillations in the X-ray case, considering their origin in the cathode stream, but under the usual experimental conditions it seems certain that the lack

of steadiness of the balance is mainly due to the instrumental side of the high tension apparatus.

How far the ionization produced by X-rays in the air of a vessel may serve as a reliable indicator of the intensity of the former is for experiment to decide. With any change of wave-length the issue becomes complicated by the selective production of characteristic and scattered radiation at any surface struck by the incident rays. The effect can hardly completely be got rid of because of the necessity for windows to the vessel. The apparatus shown in fig. 2 was constructed so as to reduce the above effect to a minimum.

Necessity for an International Unit of X-ray intensity.

The need has long been felt for a unit of X-ray intensity which shall receive International sanction. Investigators who wish to compare their results have of necessity recourse to various indicators, over the performance of which there is, however, not too much confidence. Attempts have from time to time been made to express the output of X-ray tubes in terms of the gamma rays from radium ; more, however, with a view to comparing their relative output of energy than to standardize X-ray intensity in terms of the constant source of radiation which a sealed preparation of radium affords.

The need mentioned above is also an urgent one in the practice of medical radiology at the present time. The radiologist who is dispensing X-rays wields a highly complicated collection of apparatus : he is provided with X-ray tubes, the dimensions and output of which increase almost yearly, but so far he has not been given the security he needs, namely, a unit of intensity of the rays he is using. It seems that this can hardly be done without action of some sort by physicists, and the suggestion is made here, as it has been made elsewhere as well as by others, that steps be taken to fix upon some International Unit of X-ray intensity which will serve at one and the same time the interests and requirements of physical and medical investigators.

CX. *The Measurement of Light.* By JOHN W. T. WALSH,
*M.A., M.Sc. (Department of Photometry and Illumination,
National Physical Laboratory)* *.

IN an article published under the above title in a recent
number of this Magazine †. Dr. Norman Campbell and
Mr. B. P. Dudding have criticised the logical foundation of
commonly accepted systems of definitions and in particular
the system adopted for the definitions of the principal
photometric magnitudes. Their conclusions may, perhaps,
be briefly summarized as follows :—

(1) The fundamental photometric magnitude is illumina-
tion (I).

(2) Illumination is characteristic, not of the surface
illuminated, but of the circumstances in which that
surface is placed.

(3) It can be shown experimentally that illumination is
subject to the laws of addition *if the conditions under
which the Purkinje effect operates be excluded.*

(4) On the above bases the inverse square law can be
demonstrated by suitable experiment, and, this
done, the cosine law of illumination can be demon-
strated.

(5) With the proven fact of the inverse square law
the definition of luminous intensity [candle-power]
in **a** given direction follows at once from that of
illumination ($\Phi = I r^2$)

(6) By integrating the intensity in all directions a new
quantity $F = \int \Phi \, d\omega$ is obtained (ω is solid angle).
F is characteristic of the source alone, and is termed
the flux of light from the source.

(7) Brightness, B, is defined as the intensity of unit
projected area $(\Phi/S \cos \alpha) f(\alpha)$, or normal brightness
is Φ/S where S is area.

Thus the system proposed by the authors gives the chief
photometric definitions in the order (1) illumination,
(2) luminous intensity, (3) luminous flux, and (4) brightness.
This order differs from that adopted by both the American
and the British National Illumination Committees in their
systems of definitions proposed before the International
Commission on Illumination last year ‡. It is, therefore,
desirable to examine the reason for the difference with the
object of arriving, if possible, at the most natural order.

* Communicated by the Author.
† Phil. Mag. Sept. 1922, p. 577.
‡ *Commission Internationale de l'Eclairage*, 5me Session, Paris 1921.
Rapports (in the press).

In the American definitions * the first photometric magnitude defined is luminous flux, followed by luminous intensity, illumination and brightness.

In the British definitions † the order is luminous intensity, luminous flux, illumination and brightness.

These two systems follow the respective customs of the two countries as to the rating of illuminants. In Great Britain a luminous source is rated in candles, while in America the lumen is almost universally adopted.

The existing official definitions, then, adopt as their starting point the magnitude in which the unit is realized in practice, for both luminous flux and luminous intensity are characteristic of the source alone and are independent of any other material body.

It is true that both luminous flux and luminous intensity are impossible of measurement *per se*, and it is not until the source illuminates some surface that either the flux or the intensity can be measured. It would thus appear that, in deciding on our first defined magnitude, we have to choose between (*a*) the magnitude most intimately connected with the material standard by which our unit is maintained, and (*b*) the magnitude which is directly measurable.

The authors of the paper referred to above apparently aim at the second choice, but it is difficult to understand why, after the irrefutable statement that " light measurements are based on judgments of equality of brightness of photometric surfaces," they go on without explanation to say that " the fundamental photometric magnitude is illumination."

In fact illumination appears to be the most abstruse of all the photometric magnitudes and the one of which it is most difficult to form a mental concept. It is not easy, for example, to decide whether illumination exists in the absence of a surface.

Physical photometry being disregarded, it is clear that the fundamental photometric magnitude from the point of view of measurement is "brightness." Illumination follows as that which causes surfaces to have brightness. Luminous intensity and luminous flux then follow as before.

In this connexion the proviso made by Dr. Campbell and Mr. Dudding as to the avoidance of the Purkinje effect is most interesting. On page 582 they say " But the second law of addition is not true in all circumstances ; it is not true, for example, when the Purkinje effect is apparent. For, if R_1 and R_2 are red sources, B_1 and B_2 blue sources, and if the illumination from R_1 is equal to that from B_1, and

* Illum. Eng. Soc. N.Y. Trans. xiii. (1918).
† Illuminating Engineer, London, xv. (1922).

that from R_2 to that from B_2, the illumination from R_1 and R_2 will not be always equal to that from B_1 and B_2."

This statement is most surprising, especially if taken in connexion with a previous statement that illumination is "characteristic, not of the surface illuminated, but of the circumstances in which it is placed." These two statements are not easily reconciled, for it is easily possible to imagine two pairs of photometric surfaces, one pair of high reflexion ratio and one of low reflexion ratio, so that in the former case the addition law of illumination may be found to hold, while in the latter case *with the same illuminations* it does not. Surely if the statement first quoted be true, the second is refuted.

It seems, on the whole, more consistent to regard illumination as independent both of the nature of the surface illuminated and of such essentially ocular phenomena as the Purkinje effect. The latter can, and must, be reckoned with when considering brightness, and the statement first quoted above is quite intelligible if read as referring to the addition of brightness, and if the words "brightness due to" be substituted for "illumination from."

It seems, then, that we have to choose between a system of definitions depending on (*a*) brightness, and (*b*) luminous intensity or luminous flux, as the fundamental magnitude.

It seems to the writer at least a doubtful contention that it is more logical to start with the magnitude actually perceived and measured and to work back to the magnitude in which the unit is maintained. In either case the physical laws connecting the various magnitudes in the chain have to be known and, in fact, they must be described, at any rate by implication, in the definitions of the dependent magnitudes.

The gain, if any, in logical security seems to be more than counterbalanced by a very marked loss of "concreteness" —never a pronounced characteristic of formal definitions. The mind naturally finds it most easy to form a picture of the magnitude in which the unit is maintained, that being a phenomenon having the closest association with a concrete object. In the case of photometry it would seem that the luminous intensity, or candle-power, of a source in a given direction is far more readily understood as a basis of definitions than is the brightness of a surface viewed in a given direction. For the natural physical order is (*a*) emission of luminous flux by a source owing to its luminous intensity, (*b*) incidence of this flux at a surface, (*c*) brightness of this surface due to the illumination and the power of the surface to reflect light. This is, then, the order in which the mind expects the magnitudes to be defined, and it appears to the writer the preferable order for that reason.

The function of a system of definitions is, in the writer's opinion, so to describe a number of different quantities and their relations to one another that a previous understanding of any one of these quantities (regarded as the fundamental), together with the definitions, enables all the remaining quantities to be understood also. If this be granted, the fundamental quantity should be that most generally understood. The sequence of the remaining quantities is then a matter of convenience.

In conclusion, it may not be out of place to point out that Dr. Campbell and Mr. Dudding appear not to have understood the "lambert" as a unit of brightness. It is unfortunate that they were apparently only aware of the brief statement in the Report of the Standards Committee of the Optical Society of America * that "a lambert is the brightness of a surface emitting one lumen per square centimetre of projected area in the direction considered." They naturally remark that " To speak of a lumen emitted in *a* direction is to talk nonsense."

Although, of course, this description of the lambert is quite unsound, it is unfortunate, to say the least, that the authors did not refer to the official definition of the lambert (by no means a new unit). They would then have realized that their difficulty arose, not from a miscalculation on the part of the American Committee as they seem to infer, but from an inaccuracy in the wording. " A lambert is the brightness of a perfectly diffusing surface (*i.e.* one obeying the cosine law of emission) emitting or reflecting one lumen per square centimetre " †.

Summary.

(1) The fundamental photometric magnitude from the point of view of visual measurement is brightness and not illumination.

(2) The photometric unit is one of luminous intensity (candle-power) or luminous flux.

(3) Of the two possible systems of definitions based on these respective magnitudes, that based on the magnitudes in which the unit is maintained seems preferable because it follows the natural order of mental conception.

(4) The relation between the flux unit of brightness (the lambert) and the intensity unit (the candle per square centimetre) is pointed out.

* Opt. Soc. Am. J. iv. p. 230 (1920).
† Report of Standards Committee of the Illum. Eng. Soc. N. Y. Trans. xiii. (1918).

CXI. *Notices respecting New Books.*

The Cambridge Colloquium, 1916. Part I. By G. F. EVANS. New York, 1918. Published by the American Mathematical Society.

THIS volume by Professor Griffith C. Evans of the Rice Institute consists of a course of lectures given before the American Mathematical Society at its Eighth Colloquium, held at Harvard University in 1916. The lectures dealt with the theory of functionals and their applications, and also with various other topics, including the theory of Integral Equations. The second part of the present volume is to contain the lectures of Professor Oswald Veblen of Princeton University on Analysis Situs, which were delivered at the same Colloquium.

The present lectures select for discussion the general ideas of Hadamard Stieljes, Borel, and Lebesgue in the theory of functions. It will be of great value to students to have before them in this book so clear an account of these modern developments. The work of recent writers in particular on the Lebesgue integral and on the very important development known as the Stieljes integral is summarized in excellent fashion. This volume and its companion volumes should have an excellent effect in stimulating further researches on topics which seem, indeed, to promise further rapid extension.

A Treatise on the Integral Calculus: with applications, examples, and problems. Vol. II. By JOSEPH EDWARDS. (Macmillan & Co., 1922.)

WE look on this work as our equivalent of Bertrand's treatises. The subject is developed in the good old-fashioned gentlemanly style, and the reader is not tripped up perpetually by an appeal to Rigour, Convergence, Epsilomology, and other impediments on his road—"cherchant toujours la petite bête dans la démonstration."

A summary of Elliptic Function theory finds a place, useful as a Manual for the applications encountered everywhere in Physical Science. Mention of these applications as they arise may repel the mere mathematician, but will help a reader to a grasp of general theory, as in the application of Compound Representation to discontinuous fluid motion.

Historical reference too, the valuable feature of Bertrand's style, will add to the interest, as for instance in Mercator's projection, where it appeared long after Napier's invention that Edward Wright's Table of Meridional Parts, 1599, is in reality a series of logarithmic tangents and claimed as such by Wright in his subsequent name of Nautical Logarithms.

The author does not see his way to the Continental abbreviation of the hyperbolic function, to ch, sh, th, ... in analogy with the elliptic function cn, sn, dn, tn, ...

Numerous diagrams, drawn carefully with accuracy, make a pleasing feature of the work.

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END OF THE FORTY-FOURTH VOLUME.

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